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HOLOGRAPHIC ENTANGLEMENT ENTROPY

Trabajo de investigación para obtener el grado académico de Bachiller en Ciencias con mención en Física

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Abstract

In this dissertation, we explore various aspects of holographic entanglement entropy. We start with a basic review of the topic in quantum mechanics, then we present interesting aspects of it in quantum field theory, and finally by giving a compact introduction to AdS space and conformal field theory we state the AdS/CFT correspondence that we use to understand the Ryu-Takayanagi formula. Through some non-trivial computations in the dual gravity side and by proving entropy inequalities we verify its validity. We also provide some important aspects of the RT formula such as generalizations and an heuristic derivation.

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Chapter 1

Introduction

The outline of the dissertation goes as follows: In Chapter 2, we introduce the basics of entanglement entropy in the context of quantum mechanics such as its meaning along with other types of entanglement measures that we use. In Chapter 3 we review many interesting aspects of entanglement entropy in quantum field theory for instance the relation with algebraic quantum field theory, how we can overcome the problem of divergences with mutual information and the pattern exhibited by the calculation of the entanglement entropy of a d-dimensional conformal field theory. Also we perform some explicit calculations such as the entanglement entropy for two coupled harmonic oscillators and for a for free bosons in two dimensions in the lattice. Finally, in Chapter 4 we introduce the holographic entanglement entropy proposal. We start by giving some context in the AdS/CFT correspondence and provide an heuristic way to derive the Ruy-Takayanagi formula as well as comments on its generalizations. We then provide some short calculations aimed to support the Ryu-Takayanagi proposal that were presented in the original paper and some properties obeyed in information theory that are usually hard to prove for general quantum field theories but easy in the holographic context.

Chapter 2

Entanglement in Quantum Mechanics

2.1 Composite systems

Here we talk about how to describe mathematically a system composed of two or more sub-systems. Let's suppose we have two subsystems, named A and B. They can be, for instance, two qubits: one on Mars and one on Earth. We can use basis $|i\rangle_A$ to describe the sub-system A and a basis $|j\rangle_B$ to describe the sub-system B, each one with a Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. Each sub-system can have different dimensions $d_A \neq d_B$.

The tensor product is the mathematical structure that we use to make these operations. It is a way to glue together two vectors in order to form a larger space. The tensor product between two states $|i\rangle_A$ and $|j\rangle$ is written as

$$|i, j\rangle_{AB} = |i\rangle_A \otimes |j\rangle_B \in \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}} \tag{2.1}$$

If we wish to describe the composite system $A + B$, as a basis we could use $|i, j\rangle_{AB}$. For instance, suppose A and B are spins which can be up($\uparrow$) or down($\downarrow$). Then the state $|\uparrow, \downarrow\rangle_{AB}$ means that the first system is up and the second one is down, and so on.

Now, we define the tensor structure of operators and how they distribute in states

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = (A|\psi\rangle) \otimes (B|\phi\rangle) \tag{2.2}$$

As an explicit computation, just for clearance, we have

$$\sigma_x^A |i,j\rangle_{A,B} = (\sigma_x \otimes 1_2)(|i\rangle_A \otimes |j\rangle_B) = (\sigma_x |i\rangle_A \otimes |j\rangle_B). \tag{2.3}$$

2.2 Entanglement

Quantum mechanics allows us to have more general states which are not necessarily factorisable into a product. Such linear combinations have the form

$$|\psi\rangle_{AB} = \sum_{ij} C_{ij} |i,j\rangle_{AB}, \tag{2.4}$$

where C_{ij} is a set of coefficients. If it happens that we can write $C_{ij} = f_i g_j$, then the expression above is a product state. Otherwise, it is called an *entangled state.* In other words, if $|\psi\rangle$ can be expressed as $|\psi\rangle = |A\rangle \otimes |B\rangle$ then we say that is a product state, otherwise is an entangled state. In the following subsections, we will develop the necessary machinery to give an expression that quantifies how entangled a system is.

2.3 Reduced matrix

2.3.1 Definitions

In this subsection we show the relation between mixed states and entanglement, and how this relation is a key concept to understand the relation of quantum mechanics with entanglement entropy. This connection is made by the notion of *reduced density matrix.*

Let us first recall the concept of density matrix. It is a mix of pure quantum states with classical uncertainties and is written like

$$\rho = \sum_i q_i |\psi_i\rangle \langle\psi_i| \tag{2.5}$$

where $|\psi_i\rangle$ are arbitrary states and q_i are some probabilities.

Now let's consider two sub-systems A and B and a state such as (2.4) such that $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$. If $dim(\mathcal{H}_A) = n$ and $dim(\mathcal{H}_B) = m$, the density operator would have $(n \times m)^2$ elements. For this kind of system, the density operator has information about both sub-systems and the coefficients q_i correspond to the probability over the space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. However, if the information isn't known about one of the systems, we can make a trace over one of the subspace. Thus, we define the *reduced density matrix over the system A* as

$$\rho_A = Tr_B(\rho_{AB}) = \sum_{j=1}^{m} \sum_{i}^{n\times m} q_i \langle \varphi_{B_j} | \psi \rangle \langle \psi | \varphi_{B_j} \rangle \,. \tag{2.6}$$

In a more mnemonic way for our purposes, we can write down the reduced density matrices for a bipartite system ρ_{AB} as

$$\rho_A = Tr_B \rho_{AB}, \qquad \rho_B = Tr_A \rho_{AB}. \tag{2.7}$$

The reduced density matrix allows to make measurements on $\mathcal{H}_A$ such that the result will coincide as if we were doing them in the full state that lives in $\mathcal{H}_A \otimes \mathcal{H}_B$.

$$\begin{aligned} |\psi\rangle &: \text{ separable } \iff \rho_A : \text{ pure state} \\ |\psi\rangle &: \text{ entangled } \iff \rho_A : \text{ mixed state} \end{aligned} \tag{2.8}$$

It is straightforward to observe that if the state $|\psi\rangle$ is separable, this is $\rho_{AB} = |\psi_{AB}\rangle \langle \psi_{AB}|$, then by tracing over the subspace $\mathcal{H}_B$ the density matrix ρ_A will be a pure state. The opposite happens if $|\psi\rangle$ is an entangled state because ρ_A will result in a mixed state.

2.3.2 Purification of states

If we are given a state ρ_A of a quantum system A, we can merge to it another system, which we will call R, then we can define the pure state $|AR\rangle$ such that $\rho_A = Tr_R(|AR\rangle \langle AR|)$. That is, we recover ρ_A from $|AR\rangle$ if we look at the system A alone. This procedure is called *purification.* We summarize the procedure as follows

- Use the spectral theorem to write $\rho = \sum_{j=1}^{r} \lambda_i |\phi_i\rangle \langle\phi_i|$, where $|\phi_i\rangle \in \mathcal{H_A}$

- Let's declare $\{|\chi_j\rangle\}_{j=1}^{r}$ orthonormal for Hilbert space $\mathcal{H}_B = span\{|\chi_j\rangle\}$.

  We can set
$$|\chi_j\rangle = \frac{(I \otimes \langle\phi_j|)\,|\psi\rangle}{\sqrt{\lambda_j}}$$

- Now we define $|\psi\rangle = \sum_j \sqrt{\lambda_j}\,|\phi_j\rangle \otimes |\chi_j\rangle \in \mathcal{H_A} \otimes \mathcal{H_B}$

2.4 Schmidt decomposition

2.4.1 Definitions

Using the singular value decomposition, an arbitrary pure state that belongs to the space $\mathcal{H_A} \otimes \mathcal{H_B}$ can be written in the form

$$|\psi\rangle = \sum_i \sqrt{p_i}\,|i\rangle_A\,|i\rangle_B\,, \tag{2.9}$$

where $\{|i\rangle_A\}$, $\{|i\rangle_B\}$ are orthonormal set of vectors in $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively, $\{p_i\}$ is a probability distribution and p_i are the Schmidt coefficients. Equation (2.9) is called *Schmidt decomposition*. What is interesting in this decomposition is that $|\psi\rangle$ can be expressed as a sum over only one index instead of two as we saw in equation (2.4).

The reduced states are then

$$\rho_A = \sum_i p_i\,|i_A\rangle\langle i_A|\,, \quad \rho_B = \sum_i p_i\,|i_B\rangle\langle i_B|\,, \tag{2.10}$$

where, interestingly, we observe that the coefficients of ρ_A and ρ_B coincide

With the Schmidt decomposition we can identify if the system is entangled or not by the number of Schmidt coefficients(the number of Schimdt coefficients is also called *Schmidt rank* or *Schmidt number*) that the state has in its decomposition written as in equation (2.9). If the state written in its Schmidt decomposition has only one coefficient and is different than zero, then the global state is separable. On the other hand, if it has more than one, then the global state is entangled.

Examples:

- $|\phi_1\rangle = \frac{|01\rangle+|00\rangle}{\sqrt{2}} = |0\rangle \otimes [\frac{1}{\sqrt{2}}(|1\rangle + |0\rangle]$, is a pure state.
- $|\phi_2\rangle = \frac{|00\rangle+|11\rangle+|10\rangle+|01\rangle}{2} = \frac{(|0\rangle+|1\rangle)}{\sqrt{2}} \otimes \frac{(|0\rangle+|1\rangle)}{\sqrt{2}}$, is a pure state.
- $|\phi_3\rangle = \frac{|00\rangle+|11\rangle+|01\rangle}{\sqrt{3}}$.

  Let us obtain the reduced density matrix

$$\rho_A = Tr_B(\rho_{AB}) = \frac{1}{3}(2\,|0\rangle\,\langle 0| + |0\rangle\,\langle 1| + |1\rangle\,\langle 0| + |1\rangle\,\langle 1|),$$

  where $\rho_{AB} = |\phi_3\rangle\,\langle\phi_3|$.

  Eigenvectors of ρ_A: $|\lambda_1\rangle \equiv \frac{1}{\sqrt{\frac{5+\sqrt{5}}{2}}} \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix}$ $\qquad |\lambda_2\rangle \equiv \frac{1}{\sqrt{\frac{5-\sqrt{5}}{2}}} \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix}$

  Eigenvalues of ρ_A: $\frac{3\pm\sqrt{5}}{6}$

  Then, the expansion of the reduced density matrix ρ_A in its spectral form will be:

$$\rho_A = \lambda_0\,|\lambda_0\rangle\,\langle\lambda_0| + \lambda_1\,|\lambda_1\rangle\,\langle\lambda_1|\,,$$

  and applying the procedure described in the subsection of purification of states, we obtain

$$|\chi_0\rangle = \frac{(I \otimes \langle\lambda_0|)\,|\phi\rangle}{\sqrt{\lambda_0}}\;|\chi_1\rangle = \frac{(I \otimes \langle\lambda_1|)\,|\phi\rangle}{\sqrt{\lambda_1}}$$

  Finally

$$|\phi_3\rangle = \sum_{i=0}^{1} \sqrt{\lambda_i}\,|\lambda_i\rangle\,|\chi_i\rangle\,.$$

  We can observe that $|\phi_3\rangle$ has two coefficients. Thus telling us that it is an entangled state.

2.4.2 Quantifying Entanglement

Let us have the Schmidt decomposition

$$|\psi_{AB}\rangle = \sum_{i=1}^{\text{Schmidt rank}} s_i \, |i\rangle_A \, |i\rangle_B \, , \quad s_i = \sqrt{p_i}$$

Then, $|\psi_{AB}\rangle$ is highly entangled if all the Schmidt coefficients s_i are approximately equal in magnitude, and is weakly entangled if there exists a single s_i whose magnitude is approximately 1. But we know a function that quantifies precisely this sort of behavior, this is indeed the entanglement entropy function.

2.5 Entropy

2.5.1 Von Neumann entropy

We use the entropy to measure entanglement. For this, we must make a quantum generalization of the Shannon entropy, we call this by the *Von Neumann* entropy $S(\rho)$, where ρ stands for a density operator.

$$S(\rho) = -Tr(\rho \log \rho) \tag{2.11}$$

Let's recall that density operators generalize the notion of a probability distribution. In fact, any probability distribution can be represented as a diagonal matrix. Let $\{p_i\}_{i=1}^{d}$ be a probability distribution. Then if we embed that distribution into a diagonal matrix ρ, we claim that ρ is a density matrix since

$$\sum_i \rho_{ii} = 1, \quad 0 \leq \rho_{ij} \leq 1. \tag{2.12}$$

Now, let's denote the eigenvalues of ρ by $\lambda_i(\rho)$. These eigenvalues form a probability distribution, the natural way for defining a *quantum entropy* is to apply the classical Shannon entropy to the spectrum of ρ:

$$S(\rho) \equiv H(\{\lambda_i(\rho)\}_i^d) = \sum_{i=1}^{d} -\lambda_i(\rho) \log(\lambda_i(\rho)). \tag{2.13}$$

Let's verify that both representations are the same. If $\rho = \sum \lambda_i |\psi_i\rangle \langle\psi_i|$, and applying the trace

$$\sum_k \sum_i \sum_j \langle\psi_j| \lambda_i |\psi_i\rangle \langle\psi_i| \log(\lambda_k) |\psi_k\rangle \langle\psi_k|\psi_j\rangle$$

$$\sum_k \sum_i \sum_j \lambda_i \log(\lambda_k) \langle\psi_j|\psi_i\rangle \langle\psi_i|\psi_k\rangle \langle\psi_k|\psi_j\rangle$$

$$\sum_i \lambda_i \log(\lambda_i) \tag{2.14}$$

Finally, we have that

$$S(\rho) = -Tr(\rho \log(\rho)) = -\sum_i \lambda_i \log(\lambda_i). \tag{2.15}$$

2.5.2 Some properties of the Von Neumann entropy

In the case of a biased coin flip, the Shannon entropy results in $H(X) = 0$. The quantum analogue for this behavior is $S(\rho) = 0$ iff ρ is a *pure state*. This is because we can think of a pure state as a state picked up with certainty from a mixed state. On the other hand, in the case of a fair coin flip $H(X) = 1$. This is the unique distribution maximizing H, and can be extended to quantum mechanics. Here we generalize the statement that for a density matrix ρ of dimension $d \geq 2$ whose entropy can be maximized only if $\rho = I/d$.

$$S_{max}(\rho = I/d) = \log(d) \tag{2.16}$$

Now, we present the quantum analog of independent probability distributions X and Y. The log function is chosen to ensure information is additive when two random variables are independent. Then its quantum analog will be defined as follows: Let ρ and σ be density matrices. (Entropy of a tensor product). Then, ρ and σ are independent if their joint state is $\rho \otimes \sigma$. From this it must be natural to assume the relation

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma). \tag{2.17}$$

2.5.3 Entanglement Entropy

Given a system composed of two subsystems A and B in some pure state ρ_{AB}, the entanglement entropy of A with respect to B is defined as the Von Neumman entropy of ρ_A:

$$S_{EE}(A) = S(\rho_A) = -Tr_A(\rho_A \log \rho_A), \tag{2.18}$$

and it tells us *how entangled* is the system A with the system B.

Let us recall that in the Schmidt decomposition the reduced density matrix of both systems have the same coefficients, and then

$$S_{EE}(A) = S_{EE}(B) \tag{2.19}$$

Examples:

- $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)$.

  The density matrix will be obtained with the trace operator on the Hilbert space of the other subsystem acting on the total density matrix:

  $$\rho_A = Tr_B(\rho) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

  In this case the Von Neumann entropy will be

  $$S_{EE} = -Tr(\rho \log(\rho)) = \log 2 \simeq 0.6931.$$

- $|\psi\rangle = |0\rangle \otimes [\frac{1}{\sqrt{2}}(|1\rangle + |0\rangle)]$.

  Since it is a pure state is expected that the entropy will be zero.

  $$S_{EE} = -1 \log 1 = 0.$$

- $|\psi\rangle = \sin\left(\frac{\theta}{2}\right) |10\rangle + cos\left(\frac{\theta}{2}\right) |01\rangle$.

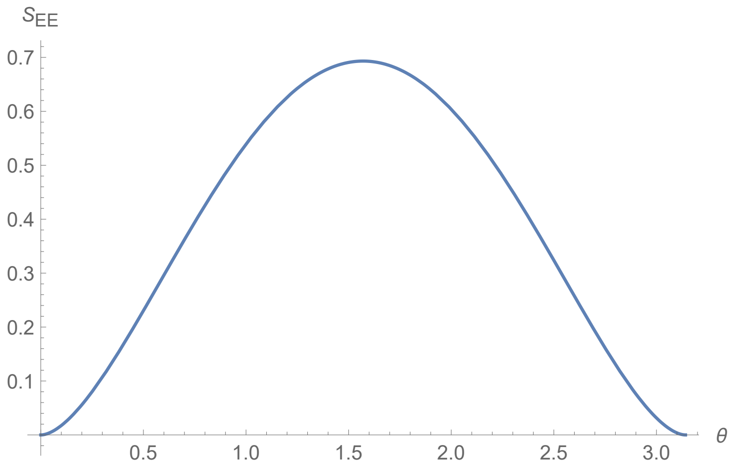


Figure 2.1: Behavior of the entanglement entropy for the state $|\psi\rangle = \sin(\frac{\theta}{2})\,|10\rangle + \cos(\frac{\theta}{2})\,|01\rangle$

The expression for the entanglement entropy for this state is

$$S_{EE} = -\cos^2\left(\frac{\theta}{2}\right)\log\left(\cos^2\left(\frac{\theta}{2}\right)\right) - \sin^2\left(\frac{\theta}{2}\right)\log\left(\sin^2\left(\frac{\theta}{2}\right)\right),$$

and it behaves as in Figure 2.1. We can observe, as discussed in the subsection of Schmidt decomposition, the closer the coefficients are the most entangled the state is. On the other hand, if one coefficient is very close to 1, then the entropy will be closer to 0 telling us that the global state is closer to be in a pure state.

2.6 Other entanglement measures

2.6.1 Relative entropy

Another important quantity in quantum information theory is *quantum relative entropy.* Given two density matrices ρ and σ, it is defined as

$$S(\rho||\sigma) = Tr(\rho\log\rho - \rho\log\sigma). \tag{2.20}$$

In classical information theory there exists the concept of relative entropy as well, defining a measure of closeness of two probability distributions. Thus, we can make an analog saying that the quantum relative entropy is a measure of closeness between the density matrices ρ and σ.

This quantity satisfies the Klein inequality:

$$S(\rho||\sigma) \geq 0, \quad S(\rho||\sigma) = 0 \text{ iff } \rho = \sigma \tag{2.21}$$

2.6.2 Subadditivity and mutual information

Consider a system AB prepared in an state ρ_{AB}. This state is in general not a product, so from the marginals $\rho_A = Tr_B(\rho_{AB})$ and $\rho_B = Tr_A(\rho_{AB})$, we can't reconstruct the original state.

$$\rho_A \otimes \rho_B \neq \rho_{AB} \tag{2.22}$$

Like in the classical case, we ask what information is contained in ρ_{AB} that is not present in $\rho_A \otimes \rho_B$. This can be defined as the distance between ρ_{AB} and the marginalized states $\rho_A \otimes \rho_B$:

$$\mathcal{I}(A:B) = S(\rho_{AB} : \rho_A \otimes \rho_B). \tag{2.23}$$

Using the definition of relative entropy:

$$S(\rho_{AB}||\rho_A \otimes \rho_B) = -S(\rho_{AB}) - Tr(\rho_{AB} \log \rho_A) - Tr(\rho_{AB} \log \rho_B). \tag{2.24}$$

To reduce the trace operations we use the partial trace in two steps. For instance, for the second term,

$$-Tr(\rho_{AB} \log \rho_A) = -Tr_A\{Tr_B(\rho_{AB}) \log \rho_A\} = -Tr_A\{\rho_A \log \rho_A\} \tag{2.25}$$

we identify that last expression as the entropy for ρ_A, i.e. $S(\rho_A)$. Doing the same for the other term, we obtain a nice way to write the mutual information in terms of entanglement entropies

$$\mathcal{I}(A:B) = S(\rho_{AB}||\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \tag{2.26}$$

The mutual information quantifies the total amount of correlations present in a state. As a byproduct of the expression of mutual information, since $\mathcal{I}(A:B) \geq 0$, we also obtain that

$$S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B). \tag{2.27}$$

This is called the *subadditivity* of the Von Neumann entropy. What this expression says that the entropy of the whole is lesser or equal than the entropy of the parts.

2.6.3 Strong subadditivity

For any three disjoint subsystems A, B and C. we have the following inequalities:

$$S_{A\cup B\cup C} + S_B \leq S_{A\cup B} + S_{B\cup C} \tag{2.28}$$

$$S_A + S_C \leq S_{A\cup B} + S_{B\cup C} \tag{2.29}$$

These are known as the *strong subadditivity*, and even though they seem straightforward, proving them is a very difficult task [1].

2.6.4 Rényi entropies

The Rényi entropies are generalizations of the Von Neumman entropy, defined for $\alpha \in [0, \infty]$. Since we are working in the quantum context let us define them directly in terms of ρ.

$$S_\alpha = \frac{1}{1-\alpha} \ln Tr(\rho^\alpha) \tag{2.30}$$

The importance of the Rényi entropies comes from the fact that $Tr(\rho^\alpha)$ is much easier to compute than $Tr(\rho \log \rho)$.

Some important properties to point out are:

- If $\alpha \to 1$ we recover the Von Neumman entropy

$$S_{\alpha\to 1} = S_1 = S \tag{2.31}$$

- It is non-negative and it vanishes when ρ is a pure state. Therefore it detects mixedness.
- The Rényi entropy detects correlations. Thus, if $I_\alpha(A : B) \neq 0$ then A and B are necessarily correlated.

$$I_\alpha(A : B) = S_\alpha(A) + S_\alpha(B) - S_\alpha(AB) \tag{2.32}$$

Example:

Let us have the reduced density matrix

$$\rho_A = \frac{1}{\alpha}[\left|0\right\rangle \left\langle 0\right| + (\alpha - 1) \left|1\right\rangle \left\langle 1\right|].$$

Then,

$$\rho_A^n = \begin{pmatrix} \left(\frac{1}{\alpha}\right)^n & 0 \\ 0 & \left(1 - \frac{1}{\alpha}\right)^n \end{pmatrix}$$

$$Tr\rho_A^n = \left(\frac{1}{\alpha}\right)^n + \left(1 - \frac{1}{\alpha}\right)^n \to S_n = \frac{1}{1-n} \log\left[\left(\frac{1}{\alpha}\right)^n + \left(1 - \frac{1}{\alpha}\right)^n\right]$$

Finally, we compute the entanglement entropy of ρ_A through the Rényi entropy setting $n \to 1$.

$$\begin{aligned} S_{n\to 1} &= \frac{1}{1-n}\Big(- \Big[\frac{\log(\alpha - 1)}{\alpha} - \log\left(1 - \frac{1}{\alpha}\right)\Big](n-1) + \mathcal{O}(n-1)^2\Big) \\ &= \frac{\log(\alpha - 1)}{\alpha} - \log\left(1 - \frac{1}{\alpha}\right) \end{aligned} \tag{2.33}$$

Where we observe that the factors of $(n - 1)$ in the numerator and the denominator cancel out one another.

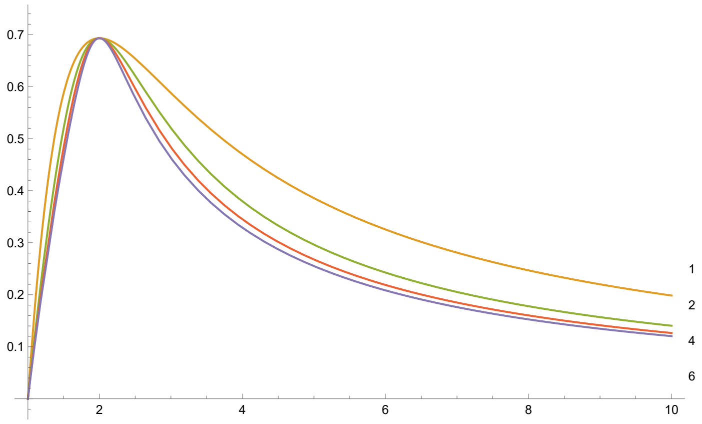


Figure 2.2: Renyi entropies for $n = \{1, 2, 4, 6\}$ and varying them with the parameter α in the x axis.

Chapter 3

Entanglement Entropy in Quantum Field Theory

In this chapter, we present some definitions and results of entanglement entropy in quantum field theory. We start by giving some context on field theory and the algebraic formulation of quantum field theory. Later, we shed some light on how this one finds a natural place when we talk about the algebras of regions and how they are related.

In the second part, we briefly explain the so-called *general structure of entanglement entropy*. That, in short, would be a structure in the organization of the terms that appear when we compute the entropy of entanglement. Some of these terms only change a bit or keep their shape even if the regularization of the theory changes.

Finally, we present some exact results that were initially found by [2] through an explicit computation of the entanglement entropy in the context of free fields using the real-time approach.

3.1 The Basics of Field Theory

3.1.1 Classical Field Theory

In field theory we are interested in a quantity called *field* which is defined at every point of space and time $(\vec{x}, t)$. Classical particle mechanics deals with finite degrees of freedom, but in field theory we turn our attention to the dynamics of fields

$$\phi(\vec{x}, t). \tag{3.1}$$

Thus, we are now dealing with infinite degrees of freedom. For a more mathematical definition one can find some definitions with fiber bundles obeying some partial differential equation, but for the purpose of this work they won't be necessary.

Example: Newtonian gravitational field

The variable here is a function $\phi(x)$, called *gravitational potential.* Then, there is a gravitational force $F(r, t)$ exerted on a mass m at spacetime position (r, t).

$$F(r, t) = -m\nabla\phi(r, t),$$

where the gravitational potential is determined by the mass distribution via the following field equation

$$\nabla^2\phi = -4\pi G\rho.$$

Example: Klein-Gordon equation

Also known as a *scalar field*, we can think of it as a function on spacetime, $\varphi : R^4 \to R$. Then the field equation is given by

$$(\nabla^2 - m^2)\varphi = 0.$$

In quantum field theory, the equation can be interpreted as the field characterizing particles with rest mass m and no other structure, namely spin, charge, etc.

3.1.2 Quantum Field Theory in a Nutshell

Though physicist do not know how to make sense of many mathematical aspects of quantum field theory, there have been efforts through *algebraic quantum field theory.* Wightman and Garding isolated certain features of quantum field theory which could be stated in mathematically precise terms. Here we give a somewhat more informal view of their axioms.

Wightman Axioms

- *Axiom 1:* The states of a quantum theory are normalised vectors in a separable Hilbert space, $\mathcal{H}$, two such that differ by a complex phase giving rise to the same state.

- *Axiom 2:* The spectrum of the energy-momentum operator P is concentrated in the closed upper light cone V^+.

$$V^+ := \{p \in R^4 : p^\mu p_\mu \geq 0, p^0 \geq 0\}$$

- *Axiom 3:* There exists in $\mathcal{H}$ a unique unit vector $|0\rangle$, which is invariant with respect to the space time translations $U(a, 1)$. We call $|0\rangle$ the vacuum state.

- *Axiom 4:* Quantum fields are operator-value distributions localized at points $x = (x^0, x^1, ..., x^{d-1})$ over the Minkowski space M_d of dimension d. The operators are defined by smearing the fields over regions acting over Hilbert spaces of states of the theory:

$$\phi(f) = \int \phi(x) f(x) d^4x,$$

  where $\phi(f)$ are called quasilocal operators. The functions f smear the operators in small regions that are not points.

- *Axiom 5:* Any two field components $\phi(f)$ and $\phi(h)$ either commute or anticommute under a space-like separation of the arguments x and y.

$$[\phi(f), \phi(h)] = 0, \quad \{\phi(f), \phi(h)\} = 0.$$

  This is a requirement to respect causality.

- *Axiom 6:* The time-slice axiom tells us that there should be a dynamical law which allows one to compute fields at an arbitrary time in terms of the fields in a small time slice

$$\mathcal{O}_{t,\epsilon} = \{x : |x^0 - t| < \epsilon\}$$

In other words, the time-slice axiom requires that given an initial condition, the dynamics is entirely determined. We can think of it as a version to quantum field theories of the evolution that dictates Schrodinger and Heisenberg equations.

- *Axiom 7:* The *covariance axiom* demands that all quantum field theories exhibit Lorentz covariance. This is,

$$U(a,\Lambda)\phi_i(x)U(a,\Lambda)^\dagger = \sum_j V_{ij}\phi_j(\Lambda x + a),$$

where V_{ij} is a complex or real matrix representation in the Poincaré group, a is a displacement in spacetime, Λ stands for the Lorentz transformation matrix and $U(a,\Lambda)$ is a unitary representation of the Poincaré spinor group.

3.2 Algebra of Operators

In the context of entanglement entropy it is useful to use the algebraic representation of operators. We are interested in how the operators of a quantum field theory organize themselves in algebras associated to spatial regions [3].

3.2.1 Algebras

An algebra of operators is a set of operators that satisfy specific rules such as linear combinations, the existence of an identity operator, etc. In symbols

$$1 \in \mathcal{A}, \quad a, b \in \mathcal{A} \to \alpha a + \beta b \in \mathcal{A}, \quad ab \in \mathcal{A}, \quad a^\dagger \in \mathcal{A}. \tag{3.2}$$

Let us define the *commutant* of a set of operators as the set of operators that commute with it

$$\mathcal{A}' = \{b; [b, a] = 0, \forall a \in \mathcal{A}\}. \tag{3.3}$$

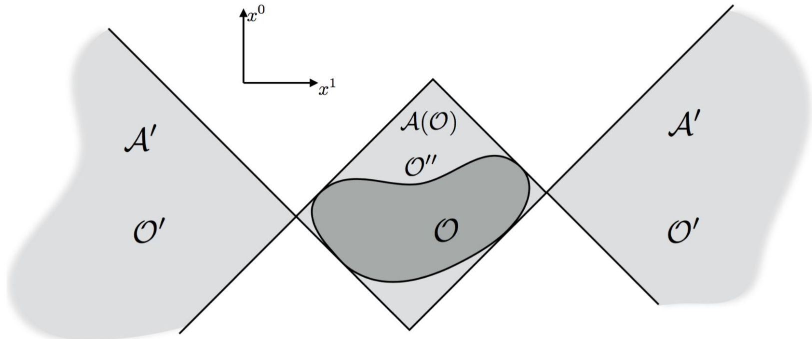


Figure 3.1: The algebra $\mathcal{A}$ is the algebra of the operators in $\mathcal{O}$ and the commutant of the algebra $\mathcal{A}$ will be $\mathcal{A}'$ corresponding to the region $\mathcal{O}'$, which is the causal complement of $\mathcal{O}$. [4]

Von Neumann theorem tells us that whatever $\mathcal{A}$ is, $\mathcal{A}'$ is an algebra, and that $\mathcal{A}$ is an algebra iff

$$\mathcal{A} = \mathcal{A}''. \tag{3.4}$$

3.2.2 Formulation in terms of algebras

We use the basic fields to associate to each open region $\mathcal{O}$ in spacetime an algebra $\mathcal{A}(\mathcal{O})$ of operators on Hilbert space,

$$\mathcal{O} \to \mathcal{A}(\mathcal{O}), \tag{3.5}$$

where $\mathcal{O}$ denotes an open region of Minkowski space. The theory is characterized by a net of algebras $\mathcal{A}$, where any $\mathcal{A}(\mathcal{O})$ algebra is generated by all $\phi(f)$ that are *smeared out* with test functions f having their support in the region $\mathcal{O}$. In addition, there are a couple of properties that the algebras must satisfy:

- *Isotony:* If $\mathcal{V} \subseteq \mathcal{O} \to \mathcal{A}(\mathcal{V}) \subseteq \mathcal{A}(\mathcal{O})$. This means that operators localized in a region have to be so in a larger one.

- *Causality:* Let $\mathcal{O}' \equiv \{x : x \text{ spacelike to } y, \forall y \in \mathcal{V}\}$. Then, $\mathcal{A}(\mathcal{O}) \subseteq (\mathcal{A}(\mathcal{O}'))'$. This property tells us that operators in spatially separated regions should commute to each other. (See figure 3.1). When the inequality becomes an equivalence, $\mathcal{A}(\mathcal{O}) = (\mathcal{A}(\mathcal{O}'))'$, the theory is said to satisfy the *Haag duality*.

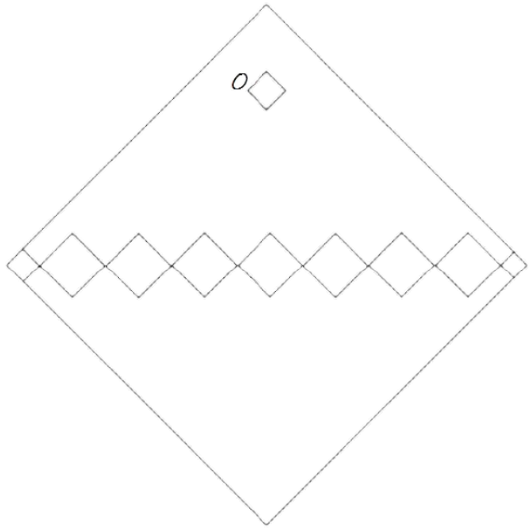


Figure 3.2: The operators near t=0 (diamond chain) will generate the algebra contained in $\mathcal{O}$. [5]

In figure 3.2, the algebra generated by the operators at region $\mathcal{O}$ is supposed to be included in the algebra generated by the operators at the thin region of diamonds at $t = 0$. This statement finds its explanation, at least heuristically, in the fact that Heisenberg operators satisfy certain causal dynamics that would allow us to determine them at $t > 0$.This is

$$\phi(x^0 + t, x^i) = e^{iHt}\phi(x)e^{-iHt}, \tag{3.6}$$

with H as the Hamiltonian of the theory. We can determine $\phi(x^0+t, x^i)$ knowing that is enough to know the field operators at the thin region of diamonds at $t = 0$ (let us call it Σ) which is inside the past light cone of $\phi(x^0 + t, x^i)$. Therefore, if we know all the algebras in Σ, we will know the algebra of all field operators of the causal complement of it.

The most natural regions have a diamond shape. These are usually known as causal regions and satisfy

$$\mathcal{O} = \mathcal{O}'' \tag{3.7}$$

3.2.3 Reeh-Schlieder Theorem

Consider a quantum field theory in the Minkowski space M with a Hilbert space H. Let Ω be the vacuum state. If we consider a small set $U \subseteq M$, there is an algebra of operators A_U acting on the vacuum state,

$$\phi(x_1)\phi(x_2)...\phi(x_n)\left|\Omega\right\rangle = A_U\left|\Omega\right\rangle, \tag{3.8}$$

where $x_i \in U$. The Reeh-Schlieder theorem states that every state in H can be approximated by $A_U\left|\Omega\right\rangle$.

Let us now discuss the physical interpretation of the theorem. Consider a time-slice of the Minkowski spacetime and a region U. Now, suppose Jupiter is in the region V, which is at a spacelike separated distance from U. Let us assign the operator J to be the creation operator of Jupiter in the region V such that

$$\left\langle\psi\right| J \left|\psi\right\rangle \approx 1 \tag{3.9}$$

for states that contain Jupiter and

$$\left\langle\psi\right| J \left|\psi\right\rangle \approx 0 \tag{3.10}$$

for states that do not. Furthermore, we know that in the vacuum state $\left\langle\Omega\right| J \left|\Omega\right\rangle \approx 0$.

What the theorem tells us is that there is an operator X in the region U that by acting on the vacuum state $\left|\Omega\right\rangle$ creates a state which contains Jupiter in V. An apparent contradiction arises because it seems that you can create Jupiter in the region V by acting on the vacuum state region even though it is causally disconnected from V

If X was a unitary operator, this would mean $XX^\dagger = 1$,

$$\left\langle X\Omega\right| J \left|X\Omega\right\rangle = \left\langle\Omega\right| X^\dagger J X \left|\Omega\right\rangle, \tag{3.11}$$

and since we established that U and V are space-separated regions, then $X^\dagger$ and J commute:

$$\left\langle X\Omega\right| J \left|X\Omega\right\rangle = \left\langle\Omega\right| J X^\dagger X \left|\Omega\right\rangle \sim 1, \tag{3.12}$$

but we also know that $\langle\Omega| J |\Omega\rangle \approx 0$, which seems to be contradictory. However, the Reeh-Schlieder theorem tells us another story. The theorem does not tell us that X could be unitary, it only claims that there exists some X in the region U that can *approximately create* the planet Jupiter in a region that is spacelike separated from U. Since the operator X cannot be unitary, this experiment cannot be implemented in a laboratory. The interpretation involves the existence of entanglement between a set U and those outside U, and the manifestation of non-local quantum correlations.

3.3 Entanglement Entropy in Quantum Field Theory

3.3.1 General Structure

The entanglement entropy is known to be divergent between adjacent regions. Thus, we need to regulate the theory to extract useful information. We can achieve this by discretizing the spacetime and taking the limit in which the spacing tends to zero. There will be some terms in the entanglement entropy which will not depend on how we regulate the theory, meaning that those terms are well defined in the continuum. These are called *universal terms.*

Asuming that our theory is scale-invariant, for a d-dimensional conformal field theory we have,

$$S^{(d)} = b_{d-2}\frac{H^{d-2}}{\delta^{d-2}} + b_{d-2}\frac{H^{d-4}}{\delta^{d-4}} + ... + \begin{cases} b_1 \frac{H}{\delta} + (-1)^{\frac{d-1}{2}} S^{\text{univ}}, & \text{(odd)} \\ b_2 \frac{H^2}{\delta^2} + (-1)^{\frac{d-2}{2}} S^{\text{univ}} \log\left(\frac{H}{\delta}\right) + b_0, & \text{(even)} \end{cases} \tag{3.13}$$

where H is a characteristic length of the region V where we compute the entanglement entropy, δ is the UV regulator, and b_i are the non-universal terms. The main difference between the expressions between odd and even dimensional is on the divergent logarithmic term that is accompanying S^{univ} in contrast with what we have for odd dimensions. The powers of (-1) are inserted by convention. The simpler case for even dimensions corresponds to a CFT in $d = (1 + 1)$ which would have the entanglement entropy $S^{(2)} = \frac{c}{3}\log\left(\frac{H}{\delta}\right) + \mathcal{O}(\delta^0)$, where c is the central charge of the CFT. Then, for odd dimensions where there are not logarithmic contributions, but instead the universal contribution goes as a constant, the case of $d = (2 + 1)$

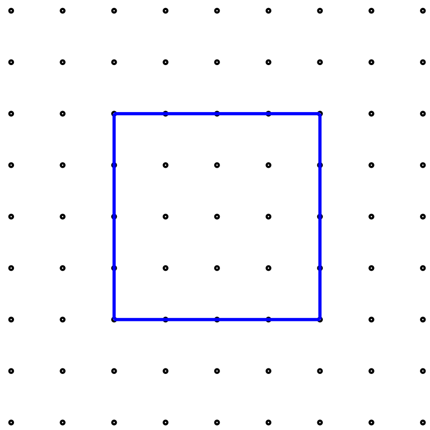


Figure 3.3: A square on the lattice

would be pictured as $S^{(3)} = b_1 \frac{H}{\delta} - F$. On the other hand, even though we call S universal, it depends on the theory and the shape, but S is *universal* in the sense that it does not depend on the choice of the regulator.

Examples on the lattice

Let us consider a model that is easy to compute, such as, for example, a free lattice model with a spacing ϵ. This ϵ will serve as a cutoff for our theory. The idea is that once our theory is discretized, and then making the lattice denser by reducing the cutoff $\epsilon \to 0$, we should obtain the same quantum field theory. By the word *same* we mean getting the same correlation functions as in the continuum limit.

We will be considering a free field and Gaussian states since from the correlation functions, we can get all the information about them. The algebra is given by the relation $[\phi_a, \pi_b] = i\delta_{ab}$, where π_b are conjugate momenta. Then, for a free field we have for a region $\mathcal{O}$ in the lattice [2]

$$S(\mathcal{O}) = Tr((C + 1/2)\log(C + 1/2) - (C - 1/2)\log(C - 1/2)), \quad C = \sqrt{XP}, \tag{3.14}$$

where X and P are the correlation matrices of the region $\mathcal{O}$,

$$X_{ab} = \langle \phi_a \phi_b \rangle|_{ab \in \mathcal{O}}, \quad P_{ab} = \langle \pi_a \pi_b \rangle|_{ab \in \mathcal{O}}. \tag{3.15}$$

In order to use (3.15), we must compute the correlation matrices. These ones will be calculated from the Hamiltonian. For a free field, with spacing 1, we will have the discretized Hamiltonian:

$$H = \frac{1}{2}\sum \pi_a^2 + \frac{1}{2}\sum_a m^2\phi_a^2 + \frac{1}{2}\sum_{ab,|x_a - x_b = 1|}(\phi_a - \phi_b)^2 = \frac{1}{2}\sum \pi_a^2 + \frac{1}{2}\sum_{ab}\phi_a K_{ab}\phi_b. \qquad (3.16)$$

The vacuum correlations on a infinite lattice are

$$\langle \phi_a \phi_b \rangle = \frac{1}{2}(K^{-\frac{1}{2}})_{ab}, \quad \langle \pi_a \pi_b \rangle = \frac{1}{2}(K^{\frac{1}{2}})_{ab}, \qquad (3.17)$$

Let us now consider the square region on the lattice as in Figure 3.3. We can see in Figure 3.4 that the entanglement entropy of this region grows linearly depending on the parameter L that is the length of the side of the square, thus obeying a relation called *area law* (in d=2+1 it would be more like a perimeter law). Although it is a bit difficult to notice on the graph, there is a subtle contribution from the logarithmic term that depends on the length L as well (see Figure 3.5).

$$\begin{aligned} S &= 0.75\frac{4L}{\epsilon} - 0.047\log\left(\frac{L}{\epsilon}\right) + \overbrace{\ldots}^{\text{contributions of lesser orders}} \\ &= 0.75\frac{\text{perimeter}}{\epsilon} - 0.047\log\left(\frac{L}{\epsilon}\right) + \ldots \end{aligned} \qquad (3.18)$$

Now we turn our attention to a squared region but rotated with respect to the lattice. The entanglement entropy for this region is

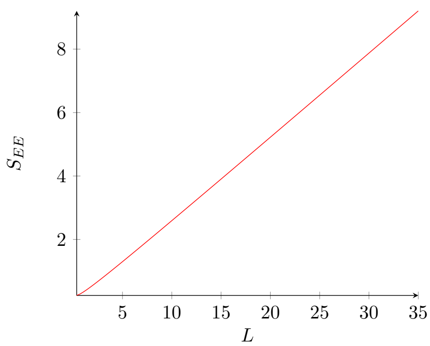


Figure 3.4: Area law

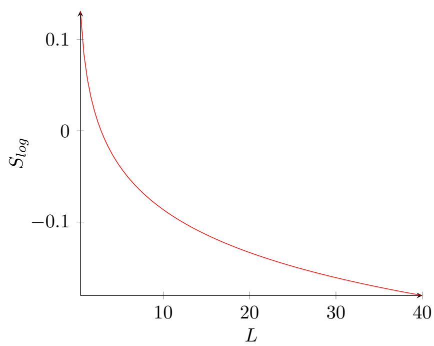


Figure 3.5: Logarithmic term

$$S = 0.85\frac{\text{perimeter}}{\epsilon} - 0.047\log(L/\epsilon) + ... \tag{3.19}$$

We observe that the area term in this last expression is different from what we got in (3.19). These terms are called *non-universal* because they depend on the way we arrive at the continuum. On the other hand, the logarithmic terms seem to hold even with those regularizations.

3.3.2 An Alternative Regulator: Mutual Information

We find divergences in the definition of entanglement entropy in quantum field theory. We can use another measurement such as the *Mutual Information* which is well defined in the continuum. According to the work made by Araki [6], this quantity can be defined without reference to entanglement entropy in terms of relative entropy.

As we saw earlier, the mutual information $I(A, B)$ between two non-intersecting regions A and B is given by

$$I(A, B) = S(A) + S(B) - S(A \cup B) \tag{3.20}$$

The mutual information is independent of any regularization used to define the underlying quantum field theory. The divergences in (3.20) present in the terms $S(A)$ and $S(B)$ are always canceled by those in $S(A \cup B)$.

We can find the entanglement entropy of a region A using mutual information. Here we present the method described in [7] in a simplified way.

The idea lies in choosing the region A of our formula (3.20) as A^- which is the reduced region A by a distance $\epsilon/2$, and the region B will be the contracted region of the complement of A' by a distance $\epsilon/2$ as well, we will call it A^+ (see Figure). This leaves an annuli region of width ϵ between the regions A^- and A^+. Since we are working with a pure quantum state, then the complement of the fence-shaped region would be $A^- \cup A^+$ and the relation between the entanglement entropies is

$$S(A, B) = S(A^- \cup A^+) = S(\text{fence}), \tag{3.21}$$

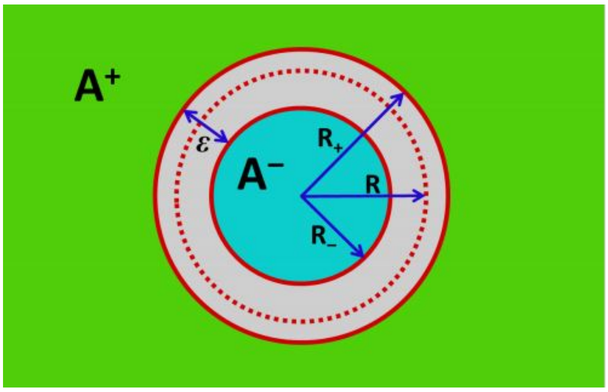


Figure 3.6: Regularization of entanglement entropy through mutual information by choosing concentric circular regions and separating them by an annuli region [7]

where ϵ is defined as normal with respect to the intermediate region in each point of it. Now, the equation (3.20) can be written as

$$I(A^-, A^+) = S(A^-) + S(A^+) - S(\text{fence}), \tag{3.22}$$

and because we can calculate all the terms, then we are able to regulate the computation of the entanglement entropy in quantum field theory. In the continuum limit of any regularization, all the UV divergences cancel each other since the divergences are local on the surface of entanglement, and the quantity $I(A^-, A^+)$ is finite.

$$S(A) = \frac{1}{2} \lim_{\epsilon \to 0} I(A^-, A^+). \tag{3.23}$$

This gives us a guide to regulate the entanglement entropy of any region without doing anything to the quantum field theory in itself. The mutual information belongs to the continuum.

$$S(A) = \frac{1}{2} \lim_{\epsilon \to 0} I(A^-, A^+). \tag{3.24}$$

3.4 Some Examples of Entanglement Entropy in Quantum Field Theory

There exist various methods to compute the entanglement entropy, to name some: replica trick, real-time method for Gaussian systems, numerical methods, etc. Unfortunately, this does not mean that it is an easy task since the computations often involve manipulating density matrices, algebras, traces, awful integrals, etc. Here we focus on the real-time method for free fields whose idea relies on relating the density matrices with the correlation functions of the fundamental fields(scalars or fermions). Once we obtain the correlation functions, we can calculate the entanglement entropy.

3.4.1 Two coupled Harmonic Oscillators

As a warm-up to introduce the real-time approach, we will be discussing the entanglement entropy for the case of two coupled oscillators. Next, we will present the case for N coupled harmonic oscillators [8][9] that can be understood as a scalar field on the lattice. Suppose the system is given by the Hamiltonian

$$H = \frac{1}{2}[p_A^2 + p_B^2 + k(x_A^2 + x_B^2) + l(x_a - x_B)^2]. \tag{3.25}$$

Through some change of variables, we can decouple the system. This is achieved by defining

$$x_\pm \equiv (x_A \pm x_B)/\sqrt{2}, \quad \omega_+ \equiv k^{1/2}, \quad \omega_- \equiv (k+2l)^{1/2} \tag{3.26}$$

Replacing them into H we get a much more tractable Hamiltonian where we are now dealing with two uncoupled harmonic oscillators and the Schrodinger equation becomes

$$\frac{1}{2}[-\partial_+^2 - \partial_-^2 + \omega_+^2 x_+^2 + \omega_-^2 x_-^2]\Psi(x_+, x_-) = E\Psi(x_+, x_-). \tag{3.27}$$

We know the ground state for a single harmonic oscillator, this is

$$\Psi_0 = \Big(\frac{m\omega}{\pi\hbar}\Big)^{\frac{1}{4}} exp\Big[\frac{-m\omega x^2}{2\hbar}\Big]. \tag{3.28}$$

If we are dealing with two systems, then the ground state will be $\Psi_{sys} = \Psi_1\Psi_2$:

$$\Psi_(x_+, x_-) = \frac{(\omega_+\omega_-)^{\frac{1}{4}}}{\pi^{\frac{1}{2}}} exp\Big[\frac{\omega_+ x_+^2 + \omega_- x_-^2}{2}\Big]. \tag{3.29}$$

Now, let us construct the reduced density matrix. Knowing that the ground state wave function is $\Psi(x_A, x_B) = (\langle x_A| \otimes \langle x_B|)\,|\Psi\rangle$, then the reduced density matrix is computed as follows:

$$\begin{aligned} \rho_A(x_A, x'_A) &= \langle x_A|\, Tr_B(|\Psi\rangle\langle\Psi|)\, |x'_A\rangle \\ &= \sum_{x_B} \underbrace{\big(\langle x_A| \otimes \langle x_B|\big)\,|\Psi\rangle}_{\Psi} \underbrace{\langle\Psi|\,\big(|x_A\rangle \otimes |x_B\rangle\big)}_{\Psi^*} \\ &= \int_{-\infty}^{+\infty} dx_B \Psi(x_A, x_B)\Psi(x'_A, x_B)^* \\ &= \sqrt{\frac{\gamma-\beta}{\pi}} exp\Big[-\frac{\gamma}{2}(x_A^2 + x_B^2) + \beta x_A x'_A\Big], \end{aligned} \tag{3.30}$$

where

$$\beta = \frac{(\omega_+ - \omega_-)^2}{4(\omega_+ + \omega_-)}, \quad \gamma = \frac{2\omega_+\omega_-}{\omega_+ + \omega_-}. \tag{3.31}$$

Finally, we need the eigenvalues of the reduced density matrix to compute the entanglement entropy. Let us first modify a bit the equation of the eigenvalue problem

$$\begin{aligned} \langle x|\,\big(\hat\rho_A\,|f_n\rangle &= p_n\,|f_n\rangle\big) \\ \langle x|\,\big(\hat\rho_A 1\,|f_n\rangle &= p_n\,\langle x|f_n\rangle\big), 1 = \int |x'\rangle\langle x|\,dx' \\ \int \langle x|\,\hat\rho_A\,|x'\rangle\langle x'|f_n\rangle\,dx' &= p_n f_n(x) \\ \int \rho_A(x, x') f_n(x') dx' &= p_n f_n(x) \end{aligned} \tag{3.32}$$

The solution for f_n is given by

$$f_n(x) = H_n(\alpha^{1/2}x)e^{\frac{-\alpha x^2}{2}}, \quad \alpha = (\omega_+\omega_-)^{\frac{1}{2}}, \tag{3.33}$$

where H_n stands for the Hermite polynomial, and for the eigenvalues

$$p_n = (1-\xi)\xi^n, \quad \xi = \frac{\beta}{\alpha+\gamma}. \tag{3.34}$$

Now, we are ready to obtain the entanglement entropy from the eigenvalues,

$$S_A = -\sum_{n=0}^{\infty} p_n \log p_n = -\log(1-\xi) - \frac{\xi}{1-\xi}\log\xi, \tag{3.35}$$

knowing that: $\xi = \frac{\beta}{\alpha+\gamma}$, $\beta = \frac{(\omega_+-\omega_-)^2}{4(\omega_++\omega_-)}$, $\gamma = \frac{2\omega_+\omega_-}{\omega_++\omega_-}$, $\alpha = (\omega_+\omega_-)^{\frac{1}{2}}$, $\omega_+ \equiv k^{1/2}$ and $\omega_- \equiv (k+2l)^{1/2}$.

3.4.2 Free Boson

Here, we compute the entanglement entropy for a free scalar for a discrete region in (1+1) dimensions. Then, we establish the relation with the correlators. The scalars and conjugate momenta obey the following commutation relations:

$$[\phi_i, \pi_j] = i\delta_{ij}, \quad [\phi_i, \phi_j] = [\pi_i, \pi_j] = 0 \tag{3.36}$$

Furthermore, let us define the correlation functions of the fields as

$$\langle\phi_i\phi_j\rangle \equiv X_{ij} \qquad \langle\pi_i\pi_j\rangle \equiv P_{ij}\langle\phi_i\pi_j\rangle = \langle\pi_i\phi_j\rangle = \frac{i}{2}\delta_{ij} \tag{3.37}$$

Normalization term

We can describe the density matrix of a reduced state through its modular Hamiltonian $H = -\log(\rho)$. Then, in terms of the modular Hamiltonian already diagonalized, we can write the density matrix as:

$$\rho_V = Ke^{-\mathcal{H}} = Ke^{-\sum_l \epsilon_l a_l a_l^\dagger}, \tag{3.38}$$

where K is a normalization constant and ϵ_l are the eigenvalues. The value of K can be obtained taking the trace of ρ_V

$$\begin{aligned} Tr(\rho_V) &= K\sum_n \prod_l \langle n| e^{-\epsilon_l a_l a_l^\dagger} |n\rangle = \sum_n \prod_l e^{-\epsilon_l n_l} \\ &= K \sum_{n_0, n_1 ...} (e^{-\epsilon_0})^{n_0} (e^{-\epsilon_1})^{n_1} (e^{-\epsilon_2})^{n_2} ... (e^{-\epsilon_l})^{n_l} \\ &= K \prod_l \Big(\sum_{n_l} (e^{\epsilon_l})^{n_l} \Big) = K \prod_l \frac{1}{1-e^{\epsilon_l}} \end{aligned} \tag{3.39}$$

Then, since we know the trace of a density operator we can finally get an expression for K

$$K = \prod_l (1-e^{\epsilon})^{-1}. \tag{3.40}$$

Entanglement Entropy from the Density Operator

Now we use the formula given by the Von Neumann Entropy $S(\rho_V) = -Tr(\rho_V \ln(\rho_V))$

$$\begin{aligned} S &= -\sum_n \langle n| Ke^{-\sum_l \epsilon_l a_l a_l^\dagger} \ln\left(Ke^{-\sum_l \epsilon_l a_l a_l^\dagger}\right) |n\rangle \\ &= -\ln(K) + K \sum_j \sum_{n_j, n_i} n_j \epsilon_j e^{-\epsilon_j n_j} \prod_{i \neq j} (e^{-\epsilon_i n_i}) \end{aligned} \tag{3.41}$$

Now we use the following result

$$\sum_{n_j} n_j \epsilon_j e^{-\epsilon_j n_j} = \frac{e^{-\epsilon_j \epsilon_j}}{(e^{\epsilon_j}-1)^2} \tag{3.42}$$

and then we replace it into (3.41) to get the entropy

$$S = -\ln(K) + K \sum_j \frac{e^{-\epsilon_j} \epsilon_j}{(e^{-\epsilon_j}-1)^2} \sum_{n_j} \prod_{i \neq j} (e^{-\epsilon_i n_i}). \tag{3.43}$$

Reordering the terms it gets the following form

$$S = \sum_l \Big(-\ln\big(1-e^{-\epsilon_l}\big) + \frac{e^{-\epsilon_l}}{1-e^{-\epsilon_l}}) \Big) \tag{3.44}$$

Knowing the eigenvalues of ρ_V, we can obtain the entanglement entropy.

Entanglement Entropy from the correlation functions

According to the work of Peschel in [10], the reduced density matrix is obtained from the properties of the correlation functions. Thus, allowing us to build up the matrix and putting it into the formula of Von Neumann entropy, and through a similar procedure as the previous subsection, we'll get an equivalent result.

Let us assume by [2] that the fields can be expressed in terms of the creation and annihilation operators

$$\phi_i = \sum_j \alpha^*_{ij} a^\dagger_j + \alpha_{ij} a_j, \quad \pi_i = \sum_j \beta^*_{ij} a^\dagger_j + \beta_{ij} a_j. \tag{3.45}$$

Then, to know more about the coefficients accompanying the fields we must use the information we know about the correlation functions. In other words, we can use a expression such as $\langle \phi_i \pi_k \rangle$ and elaborate it.

$$\begin{aligned} \langle \phi_i \pi_k \rangle &= Tr(\rho_V \phi_i \pi_j) \\ &= Tr\Big[K e^{-\sum_l a^\dagger_l a_l \epsilon_l} \Big(-i\alpha^*_{ik}\beta^*_{jm} a^\dagger_k a^\dagger_m + i\alpha^*_{ik}\beta_{jm} a^\dagger_k a_m - i\alpha_{ik}\beta^*_{jm} a_k a^\dagger_m + i\alpha_{ik}\beta_{jm} a_k a_m \Big) \Big] \end{aligned} \tag{3.46}$$

Using the value known for $\langle \phi_i \pi_k \rangle$ and reducing the trace, it finally gives us the result:

$$i\alpha_{ik} n_{kk} \beta^T_{kj} - i\alpha_{ik}(n_{kk}+1)\beta^\dagger_{kj} = \frac{i}{2}\delta_{ij}, \qquad n_{kk} = \langle a^\dagger_k a_k \rangle = \frac{1}{e^\epsilon_k - 1} \tag{3.47}$$

Doing the same procedure to the other correlators and using matrix notation, we get the following system of equations

$$\alpha^* n \beta^T - \alpha(n+1)\beta^\dagger = \frac{1}{2}\alpha^* n \alpha^T + \alpha(n+1)\alpha^\dagger = X\beta^* n \beta^T - \beta(n+1)\beta^\dagger = P \tag{3.48}$$

The equations hold if we propose $\alpha = \alpha_1 U$ and $\beta = \beta_1 U$, where U is unitary and diagonal, and α_1 and β_1 are real matrices. After replacing α and β into the system above, and then coming back to them we find the relations:

$$\alpha = -\frac{1}{2}(\beta^T)^{-1}\frac{1}{4}\alpha(2n+1)^2\alpha^{-1} = XP \tag{3.49}$$

Now, two n-by-n matrices A and B are called *similar* if there exists and invertible n-by-n matrix P such that

$$B = P^{-1}AP. \tag{3.50}$$

Then, in this sense, the matrix α is the one that makes similar to the matrices $4XP$ and $(2n+1)^2$. This results in:

$$(2n+1) = 2\alpha^{-1}\sqrt{XP}\alpha, \tag{3.51}$$

calling $C = \sqrt{XP}$ and v_k to its eigenvalues, the equation of these terms are:

$$\frac{2}{(e^{\epsilon_k}-1)} + 1 = 2v_k\epsilon_k = \ln\left(\frac{v_k+\frac{1}{2}}{v_k-\frac{1}{2}}\right) \tag{3.52}$$

Replacing ϵ_k into the last expression we gave for the entanglement entropy, we obtain

$$\begin{aligned} S &= \sum_l \Big[-\log\left(1-\frac{v_l-\frac{1}{2}}{v_l+\frac{1}{2}}\right) + \log\left(\frac{v_l+\frac{1}{2}}{v_l-\frac{1}{2}}\right)\left(\frac{v_l-\frac{1}{2}}{v_l+\frac{1}{2}}\right)\frac{1}{1-\frac{v_l-\frac{1}{2}}{v_l+\frac{1}{2}}}\Big] \\ &= \sum_l [(v_l+1/2)\log(v_l+1/2) - (v_l-1/2)\log(v_l-1/2)]. \end{aligned} \tag{3.53}$$

If we rewrite this last equation in terms of the matrix C, it results in:

$$S = Tr((C+1/2)\log(C+1/2) - (C-1/2)\log(C-1/2)), \tag{3.54}$$

which allows us to compute the entanglement entropy using only the correlators of the fields, thus resulting in a more straightforward process.

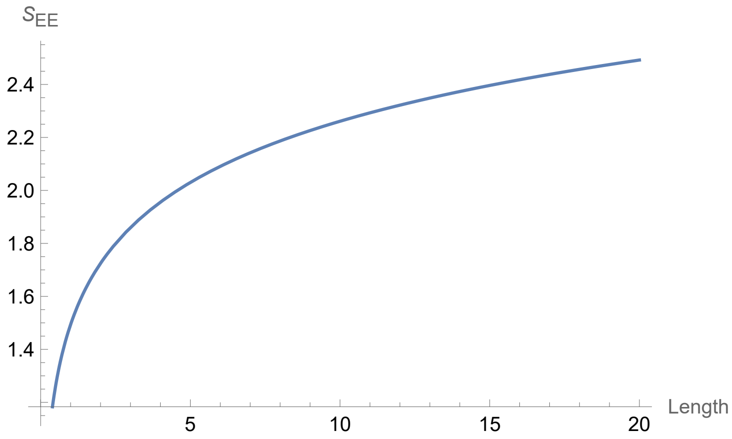


Figure 3.7: $S_{EE} = 1.493 + 0.333 \log\left(\frac{Length}{\delta}\right)$. Entanglement Entropy for a chain of fermions. As in the case for bosons, we need to find C to compute the entanglement entropy for the case of fermions, but we already know the fermionic correlators (3.59) so building C is a rather simple exercise. We can observe the logarithmic dependence with respect to the length of the chain. 0.333 is a universal constant that equals $c/3$ where c is the central charge of the theory

3.4.3 Free Fermions

Let us start with N fields satisfying the anticommutation relations $\{\psi_i, \psi_j\} = \delta_{ij}$. Then, the correlation functions are defined as:

$$\langle \psi_i \psi_j^\dagger \rangle \equiv C_{ij}, \quad \langle \psi_i \psi_j \rangle = \langle \psi_i^\dagger \psi_j^\dagger \rangle = 0. \tag{3.55}$$

Following the same steps as in the case for free bosons, we are interested in Gaussian states of the form

$$\rho_V = K e^{-\epsilon_l d_l d_l^\dagger}, \quad K \equiv (1 + e^{-\epsilon_l}), \tag{3.56}$$

where the modular Hamiltonian is already diagonalized. Similarly to the case of free bosons, we can get the ρ_V in terms of C. The result of computing the entanglement entropy which can be solely determined by the correlation functions is

$$S = -Tr((1 - C)\log(1 - C) + C\log(C)). \tag{3.57}$$

Following the procedure described in [11], and considering the Hamiltonian $-\frac{i}{2}\int dx(\psi^\dagger \partial - \partial\psi^\dagger \psi)$

in $d = (1+1)$, we can discretized it as

$$H = -\frac{i}{2} \sum_j \left[\psi_j^\dagger \psi_{j+1} - \psi_{j+1}^\dagger \psi_j \right], \tag{3.58}$$

and obtain the expression for the fermionic correlators in the lattice

$$D_{jl} = \int_0^\pi d\lambda \psi_j^{(\lambda)} \psi_l^{(\lambda)\dagger} = \begin{cases} \frac{(-1)^{(j-l)}-1}{2\pi(j-l)} & j \neq l, \\ \frac{1}{2} & j = l. \end{cases} \tag{3.59}$$

Chapter 4

Holographic Entanglement Entropy

4.1 AdS/CFT correspondence

4.1.1 The holographic principle

There must have been a reason to believe that a gravitational theory is able to describe a non-gravitational theory in one less dimension. Motivation for such duality first came from consideration of black holes, in particular from the work made by Bekenstein in 1972 [12] where he showed that the entropy of a black holes is proportional to its area.

Here we will present a somewhat heuristic discussion on the *Holographic Principle* [13]. Let us consider an isolated system of mass M_{sys} and entropy S_0 in asymptotic flat spacetime, and take A as the area of the smallest sphere that encloses the system. Now let M_A be the mass of the black hole of the same horizon area A. We must have $M_{sys} \leq M_A$. Suppose $M_{sys} < M_A$, otherwise the system would be already a black hole, now we can add $M_A - M_{sys}$ amount of energy to the system while keeping the area A fixed, then this will reach the black hole mass. Because of this process it also tells us that the black hole entropy must be greater than the initial entropy of the system.

$$S_{BH} \geq S_0 + S' \tag{4.1}$$

$$S_0 \leq S_{BH} = \frac{A}{4\hbar G_N} \tag{4.2}$$

where S' is the entropy of the added energy. This argument tells us that the maximun entropy inside a region bounded by area A is given by

$$S_{max} = \frac{A}{4\hbar G_N} \tag{4.3}$$

So now we are really treating the black holes as quantum statistical object. Now recall the definition of entropy in quantum statistical physics $S = -Tr(\rho \log \rho)$, where ρ is the density operator of the system. For a system with $N-$dimension Hilbert space $S_{max} = \log N$. Now if we compare $S_{max} = \frac{A}{4\hbar G_N}$ and $S_{max} = \log N$, then the dimension of the Hilbert space of a system inside a region of area A is bounded by

$$\log N \leq \frac{A}{4\hbar G_n} \tag{4.4}$$

So whatever we put in the area A that is the maximal entropy we may have, then if we have N degrees of freedom the maximal entropy we can have is $\log N$. Thus, it is given that the number of degrees of freedom are always bounded by the area.

The holographic principle states that in the realm of quantum gravity a region of boundary area A can be fully described by no more than $\frac{A}{4\hbar G_N}$ degrees of freedom. This is surprising because it means that we can fully describe ehat happens within a region by physical laws defined in its boundary.

4.1.2 Gravity in AdS

Basics of AdS

Let us have the Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{4.5}$$

where $R_{\mu\nu}$ is the Ricci curvature tensor, R is the Ricci scalar, $g_{\mu\nu}$ is the metric tensor, Λ is the cosmological constant and $T_{\mu\nu}$ is the energy-momentum tensor. The simplest solutions for this equation corresponds to the maximally symmetric ones, this is, with the highest degree of symmetry. If we consider a positive cosmological constant, then we would end up with the de Sitter (dS) space; considering a negative cosmological constant, then we would have the Anti de Sitter (AdS) space. However, is also well known the maxiamally symetric solution for the Einstein equation if we consider a spacetime with $T_{\mu\nu} = 0$, the Minkowski space. Here we will be interested in AdS.

We can embed the AdS_n space by considering the following line element in $(n+1)$ dimensions in a Minkowski-like spacetime $M^{n-1,2}$:

$$ds^2 = -(dX^0)^2 - (dX^1)^2 + (dX^2)^2 + ... + (dX^n)^2, \tag{4.6}$$

then we can define AdS_n space for points $(X^0, X^1, ..., X^n)$ obeying the equation:

$$X_\mu X^\mu = -(X^0)^2 - (X^1)^2 + (X^2)^2 + ... + (X^n)^2 = -a^2, \tag{4.7}$$

where a is a length scale. Now, in order to build some intuition and to keep equations short, we will be working on AdS_4. We can satisfy the embedding condition with the following coordinates:

$$X^0 = a \sin\left(\frac{\tau}{a}\right) \cosh\left(\frac{\rho}{a}\right), \quad X^1 = a \cos\left(\frac{\tau}{a}\right) \cosh\left(\frac{\rho}{a}\right), \quad \vec{X} = a \sinh\left(\frac{\rho}{a}\right) \vec{n} \tag{4.8}$$

where $\vec{X} = \{X^2, X^3, X^4\}$, $\vec{n} = \{\sin\alpha \cos\beta, \sin\alpha \sin\beta, \cos\alpha\}$. Then, we can rewrite the metric as

$$ds^2 = a^2 \left(-\cosh^2\left(\frac{\rho}{a}\right) d\tau^2 + d\rho^2 + \sinh^2\left(\frac{\rho}{a}\right) d\Omega_2^2 \right) \tag{4.9}$$

with coordinates $-\infty \leq \rho \leq \infty$, $0 \leq \tau \leq 2a\pi$, $0 \leq \alpha \leq \pi$ and $0 \leq \beta \leq 2\pi$. To describe a position in the manifold, ρ stands for a position in $\mathbb{R}$, while τ stands for a position in $\mathbb{S}$. In this way, for example, AdS_4 has the topology $\mathbb{R}^2 \times \mathbb{S}^2$ because of a bijective mapping, and representing a cylinder.

Setting $d\tau = 0$, we would be left with the metric $dl^2 = d\rho^2 + \sinh^2\left(\frac{\rho}{a}\right) d\Omega_2^2$, which represents an hyperbolic space H^3. By ignoring the singularities in $\rho = 0$, we can observe that the coordinate choice actually covers the entire manifold, and so are called the *global* *coordinates*. In general, with the latter choice of coordinates, the metric for AdS_n looks like:

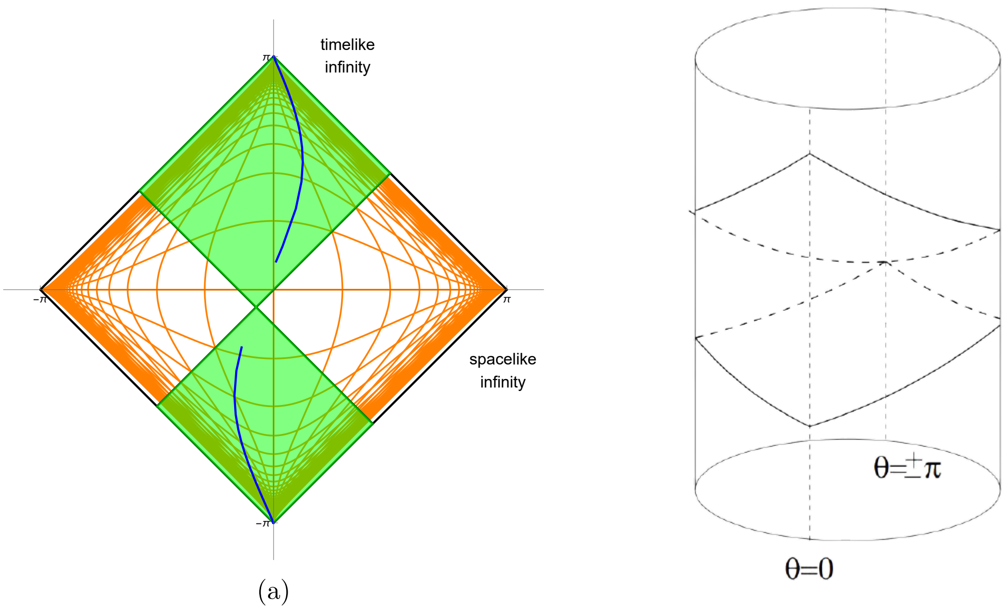
$$ds^2_{AdS_n} = a^2(-\cosh^2\rho d\tau^2 + d\rho^2 + \sinh^2\rho d\Omega^2_{n-2}) \tag{4.10}$$

In addition, we can relate the radius of the hyperboloid to the Ricci scalar through the following expressions:

$$R_{dS_n} = \frac{n(n-1)}{a^2}, \quad R_{AdS_n} = -\frac{n(n-1)}{a^2} \tag{4.11}$$

Causal Structure of AdS

AdS presents an interesting causal structure which helps on the formulation of the correspondence. It is instructive though to first present how we perform a conformal compatification on the Minkowski space. Let us have the Minkowski metric in spherical coordinates: $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$. We change variables to $u \equiv t - r$, $v \equiv t + r$ with the constraint $-\infty < u \leq v < \infty$, thus



(a) (b)

Figure 4.1: (a) Penrose diagram of the Minkowski space $M^{1,3}$. Since it is conformally related with $M^{1,3}$, the diagram must preserve null, timelike, spacelike vectors and same with geodesics. (b) Einstein static universe depicted as a cylinder where the diamond-like region is conformally related to Minkowski space.

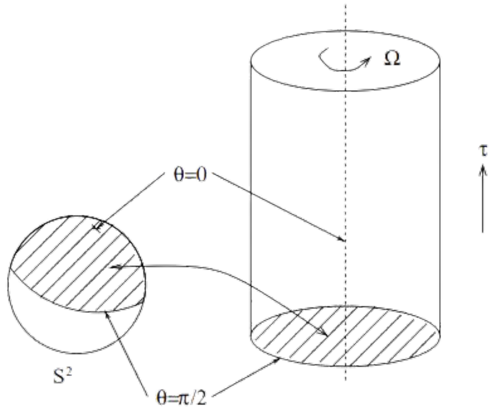


Figure 4.2: The shaded region indicates the conformal mapping between spacelike surface at constant τ and a $n-1$ dimensional hemisphere, where the equator at $\theta = \pi/2$ works as a boundary with topology $\mathbb{S}^{n-2}$, which ultimately shows the conformal mapping between AdS_n into $\mathbb{R} \times \mathbb{S}^{n-1}$.

obtaining $ds^2 = -dudv + \frac{1}{4}(v-u)^2 d\Omega^2$. Now, to compactify these coordinates, i.e. bring infinity to a finite value, we define $U = arctan(u)$ and $V = arctan(v)$ with $-\pi/2 < U \leq V < \pi/2$. Finally, we can relate conformally the Minkowski metric to part of $\mathbb{R} \times \mathbb{S}^3$ since after the last set of coordinate we can get the following metric by defining $T \equiv V + U$ and $R \equiv V - U$ with finite ranges $0 \leq R \leq \pi$ and $|T| + R < \pi$:

$$ds^2 = \omega^{-2}(-dT^2 + dR^2 + sin^2 R d\Omega^2) \tag{4.12}$$

where ω is just the conformal factor. With the coordinates now having a finite range, we can picture Minkowski space with its Penrose diagram.

Now let us look at the compactification of AdS_n. Take the metric with global coordinates, as in equation (4.9), in AdS_n setting up the length scale $a = 1$ and introducing the coordinate ψ with relation $tan\theta = sinh\rho$, where $0 \leq \theta \leq \pi/2$ we get:

$$ds^2 = \frac{a^2}{cos^2\theta}(-d\tau^2 + d\theta^2 + sin^2\theta d\Omega^2) \tag{4.13}$$

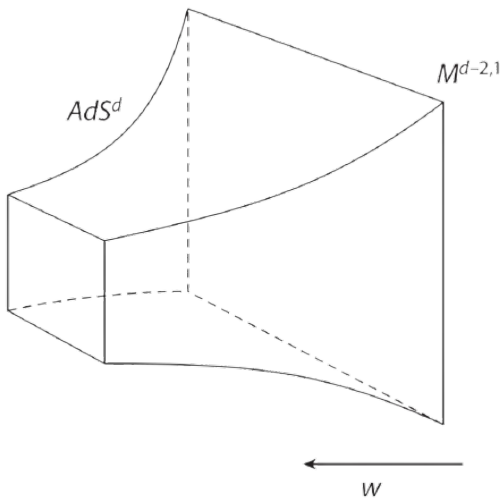


Figure 4.3: We can interpret metric (4.15) as our Minkowski metric running over w with values from 0 to ∞. As w is left as a constant, the Minkowski metric is multiplied by a factor of w, then an observer sitting at a Minkowskian slice would sees all lengths reescaled.

and by multiplying it by $a^{-2}cos^2\theta$, it becomes

$$ds'^2 = -d\tau^2 + d\theta^2 + sin^2\theta d\Omega^2. \tag{4.14}$$

We have seen this metric before, it is the metric of the conformal compactification of Minkowski space. However, the difference is on the range of θ ($0 \leq \theta \leq \pi/2$) and R ($0 \leq R \leq \pi$). This means that there must be a conformal relation between the AdS_n and the Minkowskian space of a lower dimension $n-1$, more precisely it should be a mapping of AdS_n into one half of the Einstein static universe $\mathbb{R}^{1,n-1}$.

It is important to notice that our latitude θ starts in the north pole, but never reaches the south pole, instead it only gets up to only $\pi/2$ covering this way only the northern hemisphere of S^{n-1}. This is topologically equivalent to a $(d-1)$ dimensional disk or ball B^{n-1} (Figure 4.3), showing the existence of a boundary in anti de Sitter space.

Poincaré Patch

There are many other coordinate systems that we can use to describe AdS space, some of them cover all the space while other just a wedge. Poincaré coordinates covers only a wedge but it is

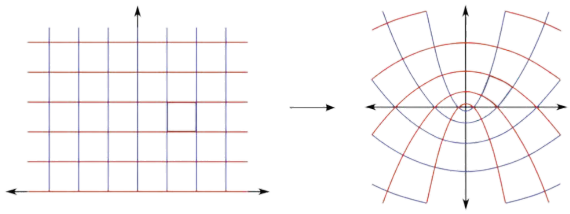


Figure 4.4: Conformal transformation preserving angles.

of great importance for the AdS/CFT correspondance. The metric for this *patch* is:

$$ds^2 = \frac{R^2}{w^2}dw^2 + \frac{w^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu \tag{4.15}$$

where $\mu = 0, ..., d-1$ and $0 \leq w < \infty$. We can find boundaries at $w = 0$ and $w \to \infty$, and if we take slices of some specific value of w, let us say w_*, then the metric becomes

$$ds^2 = \frac{w_*^2}{R^2}\eta_{\mu\nu}dx^\mu dx^\nu \tag{4.16}$$

where we can clearly see the familiarity it has with Minkowski spacetime (Figure 4.4).

4.1.3 Conformal Field Theory

A relativistic quantum field theory is invariant under the Poincaré group. In dimensions $d = 3+1$, this is the group $ISO(3,1)$. There are QFTs that are invariant under a larger set of spacetime transformations, the *conformal group*. A conformal field theory (CFT) is a QFT which is invariant under this group. CFTs are central to our understanding of the AdS/CFT correspondance as well as QFTs themselves due to its connexions with beta functions and the Renormalizatoin Group Flow (RG Flow).

The conformal group is the set of transformation that leave the metric invariant up a scale function

$$g'_{\rho\sigma}(x') = \Omega(x')g_{\rho\sigma}(x'). \tag{4.17}$$

considering an infinitesimal transformation $x'^\mu = x^\mu + \epsilon\xi^\mu(x)$ and then by taking terms of order ϵ when expanding in the ϵsmall limit, we find the conformal transformation condition amounts to

$$g_{\mu\sigma}\partial_\rho\xi^\mu + g_{\rho\nu}\partial_\sigma\xi^\nu + \xi^\lambda\partial_\lambda g_{\rho\sigma} + \kappa g_{\rho\sigma} = 0. \tag{4.18}$$

Now let us find the conformal generators for Minkowski space since this is the easiest metric with can work. This is $g_{\mu\nu} = \eta_{\mu\nu}$, with $\eta = diag(1, -1, ..., -1)$. Then (4.18) becomes:

$$\partial_\rho\xi_\sigma + \partial_\sigma\xi_\rho = \frac{2}{d}\eta_{\rho\sigma}\partial.\xi, \tag{4.19}$$

the transformations $x'^\mu = x^\mu + \epsilon\xi^\mu(x)$ generate the conformal algebra for η. If we act with $\partial^\rho = \eta^{\rho\sigma}\partial_\sigma$ on (4.19) then we get

$$d\partial^2\xi_\sigma = (2-d)\partial_\sigma(\partial.\xi). \tag{4.20}$$

We can make two conclusions:

- The case $d = 2$ is special. Any solution to the Laplace equation in (4.20) yield to a conformal transformation.
- For $d \neq 2 \rightarrow \xi^\mu$ can depend on x at most quadratically. Then all solutions can be found:

$$\xi^\mu = a^\mu + b^\mu_\nu x^\nu + cx^\mu + d_\nu(\eta^{\mu\nu}x^2 - 2x^\mu x^\nu) \tag{4.21}$$

This last expression comes in handy when we try to relate infinitesimal transformations and the generators of the respective group. We list them here for completion:

$$\begin{aligned}
P_\mu &= -i\partial_\mu \\
L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
D &= -ix^\mu\partial_\mu \\
K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu)
\end{aligned} \tag{4.22}$$

where we can identify P_μ as the generator of translations, $L_{\mu\nu}$ for Lorentz transformations, D for dilation and the so called *special conformal transformation* (SCT) that is generated by K_μ.

We can define Lorentz algebra as:

$$[J_{\mu\nu}, J_{\rho\sigma}] = -i(\eta_{\mu\rho}J_{\nu\sigma} + \eta_{\nu\sigma}J_{\mu\sigma} - \eta_{\nu\rho}J_{\mu\sigma} - \eta_{\mu\sigma}J_{\nu\rho}). \tag{4.23}$$

We can extend this algebra by adding the generator of translation $P_\mu = \partial_\mu$. Then we obtain the Poincare algebra defined by commuting

$$P_\mu = \partial_\mu, \quad \text{and} \quad J_{\mu\nu} = (x_\mu\partial_\nu - x_\nu\partial_\mu). \tag{4.24}$$

In the same way, we extend Poincare algebra by adding the dilation generator D and the special conformal generator K^μ.

$$D = x^\mu\partial_\mu, \quad \text{and} \quad K^\mu = (\eta^{\mu\nu}x^2 - 2x^\mu x^\nu)\partial_\nu, \tag{4.25}$$

in this way we finally get the conformal algebra by constructing all commutation relations. Now, let's identify the conformal algebra. A first guess comes from the number of generators, which amounts up to (P, K, D, J) : $\frac{1}{2}(d+2)(d+1)$, and that it must contain the Lorentz algebra $SO(d-1,1)$ of d-dimensional Minkowski spacetime. $SO(d,2)$ seems a good guess. We can think of the generators of the conformal algebra as J^{MN} with $M, N = 0, 1, ..., d+1$ and $\mu, \nu = 0, 1, ..., d-1$, then we can break $SO(d,2)$ into

- A matrix that's one dimension smaller. This is $n \times n \to (n-1)\times(n-1)$. This case stands for Lorentz algebra $L^{\mu\nu}$
- Two vectors with $(n-1)$ entries each. This ones stands for P^μ and K^μ, translation and conformal generators.
- One scalar. Dilation generator D.

As a rule, given $SO(d-1,1)$, the conformal algebra is $SO(d-1+1,1+1)$.

$$J_{MN} = \begin{pmatrix} J_{\mu\nu} & \frac{K_\mu - P_\mu}{2} & -\frac{K_\mu + P_\mu}{2} \\ -\frac{K_\mu - P_\mu}{2} & 0 & D \\ \frac{K_\mu + P_\mu}{2} & -D & 0 \end{pmatrix} \tag{4.26}$$

As in quantum field theory, in conformal field theory to every continuous symmetry of the action one may associate a current that is classically conserved. If the field configuration obeys classical e.o.m, the action is stationary under any field variation and

$$\partial_\mu j^\mu = 0. \tag{4.27}$$

This expression can be derived from Noether's theorem. Moreover, in order to encode the behaviour of the theory we need the *energy-momentum tensor*, it can be deduced from the variation of the action with the metric. We are interested in conformal symmetry $x^\mu \to x^\mu + \epsilon^\mu(x)$. So we have

$$j_\mu = T_{\mu\nu}\epsilon^\nu, \tag{4.28}$$

and since the current is conserved, for the case $\epsilon^\mu = const$, we obtain

$$0 = \partial^\mu j_\mu = \partial^\mu (T_{\mu\nu}\epsilon^\nu) = (\partial^\mu T_{\mu\nu})\epsilon^\nu \Rightarrow \partial^\mu T_{\mu\nu} = 0. \tag{4.29}$$

For more general transformations $\epsilon^\mu(x)$, we have $0 = \frac{1}{d} T^\mu_\mu (\partial . \epsilon)$, and since this needs to hold for arbitrary infinitesimal ϵ, we conclude that in a CFT, the energy momentum tensor is traceless:

$$T^\mu_\mu = 0 \tag{4.30}$$

Next, there is a subset of fields $\{\phi_j\} \subset \{A_i\}$ called *quasi-primary*, where $\{A_i\}$ is a set of fields. Under global transformations they transform as

$$\phi_j(x) \to \left| \frac{\partial x'}{\partial x} \right|^{\triangle_j / d} \phi_j(x'), \tag{4.31}$$

and if the theory is covariant under (4.24), then the correlation functions satisfy

$$\langle \phi_1(x_1)...\phi_n(x_n)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\triangle_1/d} ... \left|\frac{\partial x'}{\partial x}\right|_{x=x_n}^{\triangle_n/d} \langle \phi_1(x'_1)...\phi_n(x'_n)\rangle. \tag{4.32}$$

where $\triangle_j$ is the *scaling dimension*. From (4.25) it follows the expression for the two-point function:

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\triangle_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\triangle_2/d} \langle \phi_1(x'_1)\phi_2(x'_2)\rangle. \tag{4.33}$$

Translational invariance means that the N-point function doesn't not depend on N coordinates, but on their differences $(x_i - x_j)$. Furthermore, rotational invariance restricts this to $|x_i - x_j|$. So then for the two-point function we have

$$\langle \phi_1(x_1)\phi_2(x_2)\rangle = f(|x_1 - x_2|). \tag{4.34}$$

The simplest example of a CFT is a free massless scalar field,

$$\mathcal{A} = \frac{1}{2}\int d^d x \partial_\mu \phi \partial^\mu \phi \tag{4.35}$$

where in the case of $d = 4$, we have that

$$\langle \phi(x)\phi(y)\rangle = \frac{1}{|x-y|^2}. \tag{4.36}$$

4.1.4 AdS/CFT duality

The basic idea of AdS/CFT [14][15] is that it's an equality, you take the spacetime and you consider quantum gravity in that spacetime, then that should be equal to a certain quantum field theory that is defined on the boundary of that spacetime. Now we have various examples where there is a spacetime here and a specific gravity theory and a quantum field theory on the boundary. Examples involve strings theories. The known strings in the interior, in particular weakly coupled strings, which is one regime of quantum gravity then typically the gauge theory that is dual to it will be $U(N)$ gauge theory. In the large N limit, strings have a coupling

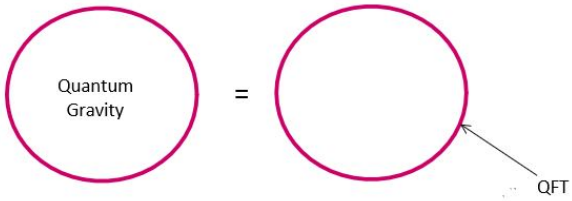


Figure 4.5: Quantum gravity "equals" to QFT on the boundary

and the coupling between the strings turns out to be proportional to $\frac{1}{N^2}$: $g' = \frac{1}{N^2}$. Now, the reason is due to an old argument of Gerard t'hooft that if you have an $U(N)$ gauge theory you can divide the Feynman diagrams of that gauge theory into planar diagrams and non planar diagrams.The latter is suppressed by parts of $\frac{1}{N^2}$, and this turns out to be the same as the division we have in string theory between string worldsheets which have the topology of the sphere of string worldsheets that have more complicated topologies which represent string interactions. Basically since that argument was suggested it was expected that large N gauge theories should be related to strings theories of some kind.

The AdS/CFT correspondence can be established as an exact relashionship between a quantum gravity that lives in asymptotically AdS_{d+1} and a CFT_d without gravity. The relation is said to be holographic since the gravitational theory lives in a higher dimension than the CFT. It can be stated as

$$Z_{grav}[\phi_0^i(x); \partial M] = \Big\langle \exp\Big(-\sum_i \int d^d x \phi_0^i(x) O^i(x) \Big) \Big\rangle_{CFT \text{ on } \partial M} . \quad (4.37)$$

where ϕ_0^i are sources and $O^i(x)$ are CFT operators. The gravitational partition function Z_{grav} comes from arguments in black holes thermodynamics. For the semiclassical regime it can be stated as:

$$Z = \int DgD\phi e^{-S_E[g,\phi]}, \quad S_E[g] = -\frac{1}{16\pi G_N} \int \sqrt{g}(R + ...) + \text{boundary terms} \quad (4.38)$$

where we are integrating over the geometry itself and the matter fields ϕ. Even though we cannot define it properly, we can use approximate it by expanding around a saddle-point:

$$Z(\beta) \approx \exp\Big(-S_E[\bar{g}, \bar{\phi}] + S^{(1)} + ...\Big). \tag{4.39}$$

On the other hand, on the left hand side, we find the familiar form of the generating functional of correlators in a CFT. We can identify there that $\phi_0^i(x)$ are the sources and $O^i(x)$ are the CFT operators. As in QFT, the correlation functions in CFT are computed with the help of partition functions and the functional operators as

$$\langle O_1(x_1)...O_n(x_n)\rangle \sim \frac{\delta^n}{\delta\phi_0^1(x_1)...\delta\phi_0^n(x_n)} Z_{CFT}[\phi_0]|_{\phi_0^i=0} \tag{4.40}$$

Probably the most well known example of the AdS/CFT correspondence comes from the original paper by Maldacena where he proposes the duality between $\mathcal{N}=4$ supersymmetryc Yang-Mills theory and type IIB string theory on $AdS_5 \times S^5$.

4.2 Ryu-Takayanagi formula(RT formula)

The RT formula is a conjecture that was proposed in 2006 by Ryu and Takayanagi in which they propose a new entry in the dictionary of the AdS/CFT correspondence to compute entanglement entropy [16]. With the RT formula [17] we would like to propose the dual gravity picture of the entanglement entropy in a CFT. In order to do this, in our CFT we start by dividing the timeslice N into two regions, namely A and B, being both complementary. Geometrically, the CFT is supposed to live at the asymptotic boundary of AdS. Let us take for example AdS_2 squashed onto a disk, then the boundary at $R=1$ is really where the CFT data "lives". Another way to see it is by considering the upper half place formulation of hyperbolic space, with coordinates $(x_1, ..., x_n > 0)$, the boundary where the CFT lives is pretty literally on $x_n = 0$. Now, if we work with Poincaré coordinates then we are setting $N = R$. To find the gravity dual we extend N in a time slice B of the bulk spacetime, then B is regarded as the hyperbolic spacetime H_{d+1}, and with this we extend ∂A to a surface γ_A in M where it is restrictred to be a co-dimension 2 which means it has to be co-dimension 1 in the timeslice in M. Then, for a CFT_{d+1} we propose

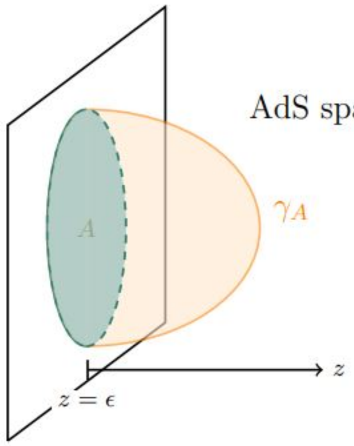


Figure 4.6: Illustration of the Ryu-Takayanagi formula.

the followinf formula:

$$S[A] = \min_{\gamma \in \Sigma} \frac{Area[\gamma]}{4G\bar{h}} \tag{4.41}$$

What this formula is telling us is the following: the entanglement entropy that is restricted to the region A is equal to the minimun over all possible surfaces that we could write down that are anchored. This follows from a stronger constraint since it has to be homologous to A. A review on the subject is given in [16].

4.2.1 Calculations that support the Ryu-Takayanagi formula

AdS$_3$/CFT$_2$: Interval in a CFT$_2$

One simple check for the RT formula is the result for the calculation for a conformal field theory in $d = 1 + 1$ of a system with certain length $l \to \infty$

$$S_A = \frac{c}{3} \log \frac{l}{a} \tag{4.42}$$

where c is the central charge, and a is the cutoff of the theory. Now we would like to calculate the very same quantity but with its gravity dual AdS$_3$. Let us first recall the Poincaré metric

$$ds^2 = \frac{L}{z^2}(-dt^2 + dx^2 + dz^2), \tag{4.43}$$

then we take a time slice $dt = 0$ to have the line element so that we can plug that into the curve equation that we finally would like to minimize to find the minimal surface *gamma*.

$$d = \int ds = \int \sqrt{\frac{L^2}{z^2}(dx^2 + dz^2)} = \int \frac{L}{Z(x)} \sqrt{1 + z(x)'} dx \tag{4.44}$$

To minimize this function we need to find a solution with the Euler-Lagrange equations $1 + Z'(x)^2 + Z(x)Z''(x) = 0$. It should gives us a function

$$Z(x) = \sqrt{(l/2)^2 - x^2} \tag{4.45}$$

We plug the result in the integral and then into the Ryu-Takayanagi formula to find the entanglement entropy for an interval $x \in [-l/2, l/2]$

$$S = \frac{2L}{4G} \int_0^{l/2} \frac{l/2}{(l/2)^2 - x^2} dx = \frac{L}{4G} \log \frac{l}{\delta} \tag{4.46}$$

This result coincides with what we got for the case of a finite interval in CFT_2. The central charge for this case is $c = 3L/(2G)$.

Spheres in a $\mathbf{CFT}_d$

For the case of spheres we follow pretty much the same method as in the case of a finite interval in CFT. First, we write down the AdS_{d+1} metric as

$$ds^2_{\mathrm{AdS}_{d+1}} = \frac{L}{z^2}(-dt^2 + dz^2 + dr^2 + r^2 d\Omega^2_{d-2}) \tag{4.47}$$

Now we want to calculate the entanglement entropy for what is inside the sphere $\mathcal{S}^{d-2}$ or radius $r = l$ and centered at $r = 0$. After considering a time slice the induced metric will be

$$ds^2 = \frac{L}{Z}\Big[(1 + Z'^2)dr^2 + r^2 d\Omega^2_{d-2}\Big], \tag{4.48}$$

then by plugging it in the Ruy-Takayanagi formula

$$S = \frac{L^{d-1}\pi^{(d-1)/2}}{2G\Gamma\left[\frac{d-1}{2}\right]} \int_0^l dr \frac{r^{d-2}}{Z^{d-1}} \sqrt{1+Z'^2}. \tag{4.49}$$

As the other example above, we minimize the expression by finding a solution to the Euler-Lagrange equation

$$rZZ'' + (d-2)ZZ'(1+Z'^2) + (d-1)r(1+Z'^2) = 0. \tag{4.50}$$

When we impose the boundary condition $Z(l) = 0$, the solution for the differential equation is $Z = \sqrt{l^2 - r^2}$. We put this solution in the RT formula

$$\begin{aligned} S &= \frac{L^{d-1}\pi^{(d-1)/2}}{2G\Gamma[(d-1)/2]} \int_{\delta/l}^1 dy \frac{(1-y^2)^{(d-3)/2}}{y^{d-1}} \\ &= \frac{L^{d-1}\pi^{(d-1)/2}}{2(d-2)G\Gamma[(d-1)/2]} \left(\frac{l}{\delta}\right)^{d-2} + \dots + \begin{cases} (-)^{\frac{d-2}{2}} 4a \log \frac{l}{\delta}, & \text{(even)} \\ (-)^{\frac{d-1}{2}} 2\pi a, & \text{(odd)} \end{cases} \end{aligned} \tag{4.51}$$

We can observe the relation to the calculation of the entanglement entropy for a d-dimensional conformal field theory we presented as an example of the general structure that emerges. Where the universal terms are $4a$ for even dimensions, and $2a\pi$ for odd dimensions.

4.2.2 Heuristic derivation

For a holographic derivation, first we have to take into account the bulk to boundary relation in AdS/CFT given by the equality of partitions functions

$$Z_{\text{String}}[\phi_0] = \left\langle e^{\int d^dx \phi_0(x^\mu)\mathcal{O}(x^\mu)} \right\rangle \tag{4.52}$$

Also, in quantum field theory, we usually use another method to compute the entanglement entropy of a region. The method is called *replica trick* and we use it as follows

$$\begin{aligned} S_A &= -\lim_{n\to 1} \frac{\log Tr_A(\rho_A^n)}{n-1} \\ &= -\lim_{n\to 1} \partial_n \log Tr_A(\rho_A^n) \end{aligned} \tag{4.53}$$

Then, we can obtain the power n of the reduced density matrix as

$$\begin{aligned} Tr_A \rho_A^q &= \frac{1}{Z_1^q} \int_{(t_E,x)\in\mathcal{R}_n} D\phi e^{-S[\phi]} \\ &= \frac{Z_q}{Z_1^q} \end{aligned} \tag{4.54}$$

where $\mathcal{R}_n$ is an n-sheeted Riemann surface and the partition function Z is the path integral over the euclidean space. Consequently, we can now calculate de Rényi entropies via

$$S_{\mathcal{A}}^q = \frac{1}{1-q} \log\left(\frac{\mathcal{Z}[\mathcal{B}_q]}{\mathcal{Z}[\mathcal{B}]^q}\right) \tag{4.55}$$

where $\mathcal{B}$ is the spacetime and $\mathcal{B}_q$ is the branched geometry we use for the replica trick. By using the AdS/CFT correspondence, we can apply the saddle point method to approximate the AdS partition function $\mathcal{Z}_{\text{String}}[\mathcal{M}_q] \approx e^{-S_{\text{Classical}}[\mathcal{M}_q]}$, in this way allows us to calculate the Rényi entropy as

$$\begin{aligned} S_A^{(q)} &= \frac{1}{1-q} \log\left(\frac{\mathcal{Z}[\mathcal{B}_q]}{\mathcal{Z}[\mathcal{B}]^q}\right) \\ &= \frac{1}{1-q} \log\left(\frac{\mathcal{Z}[\mathcal{M}_q]}{\mathcal{Z}[\mathcal{B}]^q}\right) \qquad , \\ &\approx \frac{1}{1-q}(qS_{Cl}[\mathcal{M}] - S_{Cl}[\mathcal{M}_q]) \end{aligned} \tag{4.56}$$

and if we want to calculate the entanglement entropy we have to take the limit $q \to 1$. For the replica trick we use the idea that the n^{th} power of the density matrix is the given by the partition function on the n-fold cover $\mathcal{B}_q$, this is constructed out of gluing various cut along A. Now, the parameter of the Renyi entropy, q is supposed to be running through the $\mathbb{Z}$ and so the geometric description breaks down since talking about, for example, 1.5 copies does not make sense. Is here where the work made by Lewkowycz and Maldacena [18] comes and saves us, they realized that the continuation of $q \in \mathbb{R}$ is much easier in the gravitational context.

We can exploit the feature of $\mathcal{B}$ having $\mathbb{Z}_q$ symmetry of the replica constructions. This is one the assumptions made by Lewkowycz and Maldacena's proof since that symmetry extends to the bulk replica $\mathcal{M}_q$. Then, it is convenient to consider the space $\hat{\mathcal{M}}_q \equiv \mathcal{M}_q/\mathbb{Z}_q$ which makes our Renyi entropy to simplifies into

$$S^q_{\mathcal{A}} = \frac{q}{1-q}(S_{Cl}[\mathcal{M}] - S_{Cl}[\hat{\mathcal{M}}_q]) \tag{4.57}$$

This result matches pretty well with what we were expecting as in principle we could have rewritten the replica trick of entanglement entropy in terms of the bulk action

$$S_A = \lim_{n\to 1} \partial_n (S_{\text{bulk}}[\mathcal{B}] - nS_{\text{bulk}}[\mathcal{B}]) \tag{4.58}$$

Now, going back to the n-sheeted surface $\mathcal{R}$, this can be characterized by a deficit angle δ on the surface ∂A. This causes the Ricci scalar to behave like a delta function

$$R = 4\pi(1-n)\delta(\gamma_A) + R^0, \tag{4.59}$$

then by plugging this expression in the Einstein-Hilbert action for the Anti de Sitter theory we have

$$\begin{aligned} S_{AdS} &= -\frac{1}{16\pi G_{d+2}} \int_M dx^{d+2}\sqrt{g}(R+\Lambda) + ... \\ &= -\frac{1}{4G_{d+2}} \int_M dx^{d+2}\sqrt{g}\delta(\gamma_A) + ... \\ &= -\frac{(1-n)\text{Area}(\gamma_A)}{4G_{d+2}} + ... \end{aligned} \tag{4.60}$$

Now we use the bulk to boundary relation to calculate holographically the entanglement entropy

$$
\begin{aligned}
S_A &= -\frac{\partial}{\partial n} \ln Tr\rho_A^n \Big|_{n=1} \\
&= -\frac{\partial}{\partial n} \ln Z_{CFT} \Big|_{n=1} \\
&= -\frac{\partial}{\partial n} \ln e^{-S_{AdS}} \Big|_{n=1} \\
&= \frac{\partial S_{AdS}}{\partial n} \Big|_{n=1} = \frac{\partial}{\partial n} \Big[\frac{(n-1)\text{Area}(\gamma_A)}{4G_{d+2}} \Big]_{n=1} = \frac{\text{Area}(\gamma_A)}{4G_{d+2}}
\end{aligned}
\tag{4.61}
$$

Thus, giving the same exact result of the Ryu-Takayanagi formula proposed earlier.

4.2.3 Generalizations

Quantum corrections

The Ryu-Takayanagi formula needs quantum corrections like many other formulas and we learned the result of such correction from Faulkner, Maldacena and Lewcowycz [19]. There exist quantum corrections given by the bulk entanglement entropy. Let us consider the boundary region A, then the minimal surface will divide the bulk into two pieces, the entanglement entropy between these two regions is what gives the quantum corrections.

$$
S_B = \underset{\mathcal{X}}{min} \Big[\frac{A(\mathcal{X})}{4G} + S_{\text{bulk}}(\mathcal{X}) \Big] \tag{4.62}
$$

It can be derived from *gravitational path integral*, it is our most reliable source of information of quantum gravity and it has led to a number of successes including a derivation of the *Page curve* for an evaporating black hole.

Stringy corrections

In this case we would like to have a more general formula for entanglement entropy in duals with higher derivative gravity. This is, instead of just consider Einstein gravity we will consider Einstein gravity with higher curvature terms, as a result these terms appear as corrections in α' which can be regarded as stringy corrections.

We are looking for something that replaces the area in the Ryu-Takayanagi formula. This is

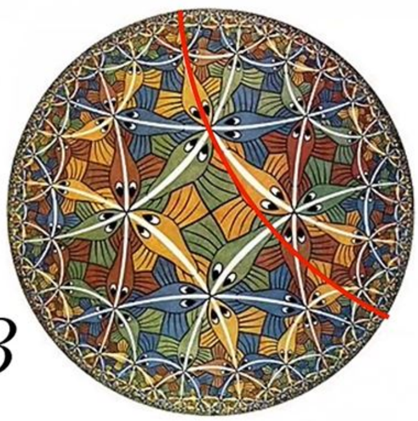


Figure 4.7: The entanglement entropy between the regions separated by the red mark causes the quantum corrections.

rather analogous to the Bekenstein-Hawking entropy in the case of black holes is generalized to the Wald entropy in the case of higher derivative gravity where this formula is

$$S_{Wald} = -2\pi \int d^d y \sqrt{g} \frac{\partial L}{\partial R_{\mu\rho\nu\sigma}} \epsilon_{\mu\rho}\epsilon_{\nu\sigma} \tag{4.63}$$

where we are taking a single derivative of the Lagrangian with respect to the Riemann tensor contracted with the epsilon tensors. Consider a general Lagrangian built from contractions of Riemann tensors

$$S = \int d^d x \sqrt{D} L(R_{\mu\rho\nu\sigma}), \tag{4.64}$$

then Xi Dong proposed for the entanglement entropy the following generalization for higher derivative gravity:

$$S_{EE} = 2\pi \int d^d y \sqrt{g} \Bigg\{ \underbrace{\frac{\partial L}{\partial R_{z\bar{z}z\bar{z}}}}_{\text{Wald's formula}} + \underbrace{\sum_{\alpha} \Big(\frac{\partial^2 L}{\partial R_{zizj} \partial R_{\bar{z}k\bar{z}l}} \frac{8 K_{zij} K_{\hat{z}kl}}{q_\alpha + 1} \Big)_\alpha}_{\text{Anomaly from extrinsic curvature}} \Bigg\} \tag{4.65}$$

where $K_{...}$ denotes the extrinsic curvature tensor of a codimension-2 surface.
As an example, for $f(R)$ gravity with Lagrangian $L = \frac{d(d-1)}{L^2} + R + f(R)$ the entanglement entropy can be read as

$$S^{f(R)} = \frac{A(V)}{4G} + \frac{1}{4G} \int_V d^{d-1} y \sqrt{h} f'(R), \tag{4.66}$$

where the first term corresponds to the Ryu-Takayanagi formula, and the second term stands for the correction that can be understood as well as some kind of area which depends on $f(R)$. For a full explanation of the derivation of such corrections refer to [20] [21].

4.2.4 Properties of holographic entanglement entropy

Strong subadditivity

When we divide a quantum system into regions, they might overlap or be disjoint, certain inequalities properties hold. For example, the strong subadditivty inequality reads as follows

$$S(A) + S(B) \geq S(A \cup B) + S(A \cap B) \tag{4.67}$$

Now if we take the system to be composed of more than two subsystems, $\mathcal{H}_{full} = \otimes_i \mathcal{H}_i$, then the subadditivity inequality can be strengthened to

$$S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_2) + S(\rho_{123}), \quad S(\rho_{12}) + S(\rho_{23}) \geq S(\rho_1) + S(\rho_3) \tag{4.68}$$

This property is known as *strong subadditivity* [22]. Its proof only considering aspects of information theory is highly non trivial [1] [23], but with holography we can circumvent this and find a very simple proof that we will present here.

For regions A and B let us define its corresponding minimal hypersurface $m(A)$ and $m(B)$ ending in ∂A and ∂B respectively. Now, let us define the regions

$$r_{A\cup B} = r_A \cup r_B, \quad r_{A\cap B} = r_A \cap r_B, \tag{4.69}$$

along with its boundaries

$$\partial r_{A\cup B} = (A \cup B) \cup m_{A\cup B} \quad \partial r_{A\cap B} = (A \cap B) \cup m_{A\cap B}. \tag{4.70}$$

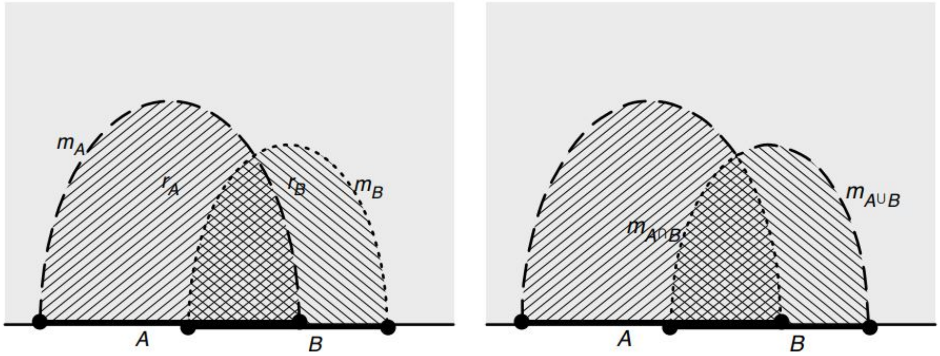


Figure 4.8: Two overlapping regions A and B with their respective minimal hypersurface m_A and m_B, on the left. And on the right, $m_{A \cap B}$ and $m_{A \cup B}$ are rearrangements of the original minimal hypersurfaces, though they do not represent necesarilly the minimal hypersurfaces of regions $A \cap B$ and $A \cup B$ respectively.

The hypersurface $m(A \cup B)$ is not necesarilly the the minimal hypersurface of $A \cup B$, but it works more as an upperbound. Then we have

$$m(A \cup B) \geq min\{m(A \cup B)\}, \tag{4.71}$$

same with $A \cap B$

$$m(A \cap B) \geq min\{m(A \cap B)\} \tag{4.72}$$

. Both inequalities imply

$$m(A \cup B) + m(A \cap B) \geq min\{m(A \cup B)\} + min\{m(A \cap B)\}. \tag{4.73}$$

From figure [x] we notice that $m_{A \cup B} \cup m_{A \cap B} = m_A \cup m_B$, combining this with a reescale of the prior equation for a factor of $4G_N$ we get the relation

$$a(m_{A \cup B}) + a(m_{A \cap B}) = a(m_A) + a(m_B) \tag{4.74}$$

that completes the proof.

Monogamy of mutual information

Let us recall the definition of the mutual information:

$$I(A:B) = S(A) + S(B) - S(AB) \tag{4.75}$$

In the section of quantum field theory, we have explained that this quantity is useful because it cancels ultraviolet divergences in the entanglement entropy and it serves as an upper bound on two-point correlation functions. In particular we can mention the quantum Pinsker inequality

$$\left(\frac{\langle \mathcal{O}_A \mathcal{O}_B \rangle - \langle \mathcal{O}_A \rangle \langle \mathcal{O}_B \rangle}{\|\mathcal{O}_A\| \|\mathcal{O}_B\|} \right) \leq 2I(A:B) \tag{4.76}$$

that tells us that if we try to evaluate the connected correlator there is an upper bound on how large it can be, and is bounded by the mutual information.

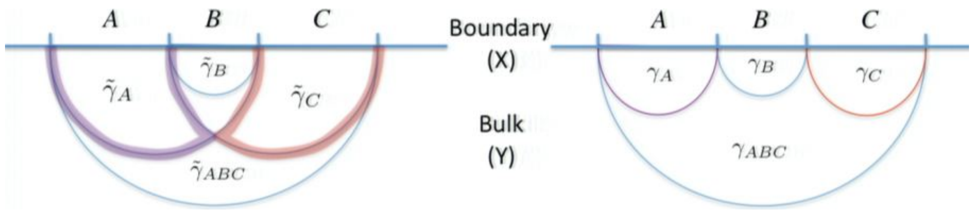


Figure 4.9: In both illustrations, the horizontal line stands for the boundary CFT and what comes below is the bulk. On the right, we depict the minimal surfaces following the regions inside the functions in which we want to compute the entanglement entropy $S(A) + S(B) + S(C) + (ABC)$, and on the left, we follow $S(AB) + S(BC) + S(AC)$.

Now, let us focus more on this holographic field theories with the Ryu-Takayanagi prescription and taking into account the I_3 mutual information function

$$I_3(A:B:C) := S(A) + S(B) + S(C) - S(AB) - S(BC) - S(AC) + S(ABC). \tag{4.77}$$

Looking at the figure above, now we will try to proof that the left is as least as much as the right hand side. As in the case of strong subadditivity, we will cut up the surfaces on the left and reorganize them. The surface on purple will be called γ'_A and the one in orange will be γ'_B, and these surfaces need not be minimal surfaces. Reorganizing the areas associated with the sum of the entropies in AB, BC and AC, into some other surfaces that satisfy the same hypothesis that is required by

$$S(AB) + S(BC) + S(AC) \geq S(A) + S(B) + S(C) + S(ABC) \tag{4.78}$$

without necessarily being minimal. This result is often called *superextensivity*.

$$\begin{aligned} S(AB) + S(BC) &\geq S(ABC) + S(B) + [S(A) + S(C) - S(AC)] \\ &= S(ABC) + S(B) + I(A:C) \end{aligned} \tag{4.79}$$

Also, it serves as a check of the Ryu-Takayanagi conjecture since it is consistent with all known inequalities satisfied by the Von Neumann entropy.

If we look at the four-partite information, it should look something like

$$I_4(A:B:C:D) = -\sum_{J\subset\{A,B,C,D\}} (-1)^{|J|} S(J) \tag{4.80}$$

and the naive proof for $I_4(A:B:C:D) < 0$ seems hold, but we can actually find some counterexamples in, for example, a $(1+1)$ dimensional CFT. This motivates being more careful with the proof since we are also interested in working with higher dimensional systems other than the one shown in the previous figure which can be misleading since it is a very specific configuration.

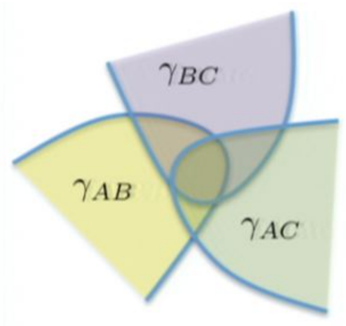


Figure 4.10: For a much complete proof, we break $\partial_Y \gamma_{AB}$ into four pieces.

Taking some inspiration from the previous approach, we will cut up and reorganize chunks of the minimal surface, but this time for regions AB, BC and AC. For example, for AB we can break its bulk surface(we will use Y to label things belonging to the bulk) as

$$
\begin{aligned}
&\partial_Y \gamma_{AB} \cap (\gamma_{BC} \setminus \gamma_{AC}) \\
&\partial_Y \gamma_{AB} \cap (\gamma_{BC} \cup \gamma_{AC}) \\
&\partial_Y \gamma_{AB} \cap (\gamma_{AC} \setminus \gamma_{BC}) \\
&\partial_Y \gamma_{AB} \setminus (\gamma_{AC} \cup \gamma_{BC})
\end{aligned} \tag{4.81}
$$

and we do the same for $\partial_Y \gamma_{BC}$ and $\partial_Y \gamma_{AC}$, so we would end up with twelve different pieces and then we put the parts together by defining bulk regions (see Figure x).

$$
\begin{aligned}
\tilde{\gamma}_A &:= \gamma_{AB} \cap \gamma_{AC} \setminus \gamma_{BC} \\
\tilde{\gamma}_B &:= \gamma_{AB} \cap \gamma_{BC} \setminus \gamma_{AC} \\
\tilde{\gamma}_C &:= \gamma_{AC} \cap \gamma_{BC} \setminus \gamma_{AB} \\
\tilde{\gamma}_{ABC} &:= \gamma_{AB} \cup \gamma_{BC} \cup \gamma_{AC}
\end{aligned} \tag{4.82}
$$

Now we can ask if with this new definitions for the regions they actually end where they are supposed to, but if we ask about its pieces associated with bulk we can observe it has three parts regarding region A

$$
\begin{aligned}
\partial_Y \tilde{\gamma}_A &= \partial_Y (\gamma_{AB} \cap \gamma_{AC} \setminus \gamma_{BC}) \\
&= (\partial_Y \gamma_{AB} \cap \gamma_{AC} \setminus \gamma_{BC}) \cup (\gamma_{AB} \cap \partial_Y \gamma_{AC} \setminus \gamma_{BC}) \cup (\gamma_{AB} \cap \gamma_{AC} \cap \partial_Y \gamma_{BC}),
\end{aligned} \tag{4.83}
$$

and the same happens with the four regions we have just defined, then if we write them down they match up perfectly. In this way we can proof more rigorously the superextensive property and at the same time proving that

$$
I(A : B : C) \leq 0 \tag{4.84}
$$

which is called the *monogamy of entanglement*. The holographic proof for this property was first developed in [24].

In information theory, monogamy can be understood by considering three quantum subsystems and that the more entangled A is with B, the less entangled A can be with C. If we take the extreme case where A is in a pure entangled state with B, then that state should be pure with C

$$|\psi\rangle_{ABC} = |\phi\rangle_{AB} \otimes |\omega\rangle_C \,. \tag{4.85}$$

This manifests quantitatively by *monogamy relations* which are just superextensivity statements. Monogamy suggests mutual information is detecting entanglement.

Chapter 5

Conclusions

Now, let us take a step back and appreciate what we have discussed. To start, we introduced some basic machinery in quantum mechanics to understand more intuitively what is entanglement entropy and how we can compute it. It was presented some other ways to measure how entangled quantum states are that appeared in other sections and were of great importance in overcome infinities and provide other ways to compute entanglement entropy in quantum field theory. We then explored many aspects of entanglement entropy in quantum field theory including some explicit calculations that shed some light into its interpretation and as how difficult it is to obtain results analytically. On the way we got to study some non-trivial aspects such the Reeh-Schlieder theorem, the apparent pattern of general structure in conformal field theories and how algebra finds naturally a place when we talk about entanglement in quantum field theory. Next, we stated the AdS/CFT correspondence along with important aspects to consider from both theories, Anti de Sitter spacetimes and conformal field theories. This allowed a somewhat more precise presentation of the bulk to boundary relation. Though, the original statement came from Maldacena with the relation of string theories in a AdS background with a conformal field theory called super Yang Mills theory.

After having developed all the ingredients, we presented the Ryu-Takayanagi prescription of holographic entanglement entropy. Through the presentation of explicit calculations we observed how it connects field theory states with purely geometric objects. We then provided a heuristic derivation of the proposal using the Maldacena conjecture and another method to compute entanglement entropy called the replica trick. We then gave a holographic proof of two inequalities, the strong subadditivity and monogamy of mutual information, both of them short and elegant compared to the original proofs given in the context of information theory and quantum field theory.

As a summary of the whole dissertation, we found that the entanglement entropy of a region of a field theory is directly proportional to the area of a region bounded by a minimal surface co-dimension 2 that is anchored to the boundary, this is

$$S_A \propto Area(\gamma_A) \tag{5.1}$$

What is exciting of this little relation is its deep relation with quantum dynamics and geometry. It may help to give some answers regarding the very fabric of spacetime and more deep questions about the AdS/CFT correspondence and thus allowing us to know more about quantum gravity.

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